1/11

# Path Integrals in Quantum Mechanics

Consider a quantum system described by coordinates $q_i$, conjugate momenta $p_i$, Hamiltonian $H(q_i, p_i)$.

Transition amplitude from state $|q_a\rangle$ to state $|q_b\rangle$ after some time $T$:

$$U(q_a, q_b, T) = \langle q_b | e^{-iHT} | q_a \rangle$$

Divide $T$ into $N$ intervals of time $\epsilon = \frac{T}{N}$.

$$e^{-iHT} = e^{-iH\epsilon} e^{-iH\epsilon} \cdots e^{-iH\epsilon}$$

Insert complete sets of states $\int \prod_i dq_1^i |q_1\rangle\langle q_1| = 1$.

$$e^{-iHT} = e^{-iH\epsilon} \int \prod_i dq_1^i |q_1\rangle\langle q_1| e^{-iH\epsilon} \int \prod_j dq_2^j |q_2\rangle\langle q_2| e^{-iH\epsilon} \cdots$$

Take the limit $\epsilon \to 0$: $\langle q_{k+1} | e^{-iH\epsilon} | q_k \rangle \to \langle q_{k+1} | (1 - iH\epsilon) | q_k \rangle$

Consider $\langle q_{k+1} | f(q) | q_k \rangle = f(q_k) \prod_i \delta(q_k^i - q_{k+1}^i)$

$$= f\left(\frac{q_k + q_{k+1}}{2}\right) \left(\prod_i \int \frac{dp_k^i}{2\pi}\right) \exp\left[i \sum_i p_k^i (q_{k+1}^i - q_k^i)\right]$$

Consider $\langle q_{k+1} | f(p) | q_k \rangle = \left(\prod_i \int \frac{dp_k^i}{2\pi}\right) \langle q_{k+1} | f(p) | p_k \rangle \langle p_k | q_k \rangle$

$$= \left(\prod_i \int \frac{dp_k^i}{2\pi}\right) f(p_k) \exp\left[i \sum_i p_k^i (q_{k+1}^i - q_k^i)\right]$$

If $H$ does not contain any terms which mix $q$ and $p$,

$$\langle q_{k+1} | H(q,p) | q_k \rangle = \left( \prod_i \int \frac{dp_k^i}{2\pi} \right) H\left( \frac{q_{k+1}+q_k}{2}, p_k \right) \exp\left[ i \sum_i p_k^i (q_{k+1}^i - q_k^i) \right]$$

Since $e^{-iH\epsilon} \approx 1 - iH\epsilon$,

$$\langle q_{k+1} | e^{-iH\epsilon} | q_k \rangle = \left( \prod_i \int \frac{dp_k^i}{2\pi} \right) \exp\left[ -iH\left( \frac{q_{k+1}+q_k}{2}, p_k \right) \epsilon \right]$$
$$\times \exp\left[ i \sum_i p_k^i (q_{k+1}^i - q_k^i) \right]$$

Define $q_0 = q_a$, $q_N = q_b$. The transition amplitude is

$$\boxed{U(q_a, q_b, T) = \left( \prod_{i,k} \int dq_k^i \int \frac{dp_k^i}{2\pi} \right) \exp\left[ i \sum_k \left( \sum_i p_k^i (q_{k+1}^i - q_k^i) - \epsilon H\left( \frac{q_{k+1}+q_k}{2}, p_k \right) \right) \right]}$$

As $\epsilon \to 0$, $\epsilon \sum_k \to \int_0^T dt$

$$\frac{q_{k+1}+q_k}{2} \to q(t) \quad , \quad p_k \to p(t)$$

$$p_k \frac{(q_{k+1} - q_k)}{\epsilon} \to p \cdot \dot{q}(t) = p \cdot \frac{dq}{dt}$$

$$\prod_{i,k} \int dq_k^i \to \int \mathcal{D}q(t) \quad , \quad \prod_{i,k} dp_k^i \to \int \mathcal{D}p(t)$$

$$\boxed{U(q_a, q_b, T) = \int \mathcal{D}q(t) \mathcal{D}p(t) \exp\left[ i \int_0^T dt \left( \sum_i p^i \dot{q}^i - H(q^i, p^i) \right) \right]}$$

If $H = \frac{p^2}{2m} + V(q)$ then we can do the p-integrals:

$$\int \frac{dp_k}{2\pi} \exp\left[i\left(p_k(q_{k+1}-q_k) - \epsilon \frac{p_k^2}{2m}\right)\right] = \underbrace{\sqrt{\frac{-im}{2\pi\epsilon}}}_{\equiv 1/C(\epsilon)} \exp\left[\frac{im}{2\epsilon}(q_{k+1}-q_k)^2\right]$$

Then,

$$U(q_a, q_b, T) = \frac{1}{C(\epsilon)}\left(\prod_k \int \frac{dq_k}{C(\epsilon)}\right) \exp\left[i\sum_k \epsilon\left(\frac{m}{2}\frac{(q_{k+1}-q_k)^2}{\epsilon^2} - V\left(\frac{q_{k+1}+q_k}{2}\right)\right)\right]$$

As $\epsilon \to 0$ we arrive at the Feynman Path Integral

$$\boxed{U(q_a, q_b, T) = \int \mathcal{D}q(t) \exp\left[i\int_0^T dt\, L[q, \dot{q}]\right]}$$

where $L[q, \dot{q}] = \frac{m}{2}\dot{q}^2 - V(q)$

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Functional Integral Quantization of Scalar Fields

Replace $q^i \longrightarrow \phi(\vec{x})$

$$H = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi)\right]$$

$$L = \int d^3x\, \mathcal{L} = \int d^3x \left[\frac{1}{2}(\partial_\mu \phi)^2 - V(\phi)\right]$$

$$\boxed{\langle \phi_b(\vec{x}) | e^{-iHT} | \phi_a(\vec{x}) \rangle = \int_{\boxed{\substack{\phi(0)=\phi_a \\ \phi(T)=\phi_b}}} \mathcal{D}\phi \exp\left[i\int_0^T d^4x\, \mathcal{L}\right]}$$

Note that except for the dependence on T, the final expression for the transition amplitude is Lorentz invariant.

## Time-Ordered Expectation Values

Consider

$$\int_{\substack{\phi(-T)=\phi_a \\ \phi(T)=\phi_b}} \mathcal{D}\phi(x)\ \phi(x_1)\,\phi(x_2)\ \exp\left[i\int_{-T}^{T} d^4x\ \mathcal{L}\right]$$

$$= \int \mathcal{D}\phi_1(\vec{x}) \int \mathcal{D}\phi_2(\vec{x}) \int_{\substack{\phi(t_1)=\phi_1 \\ \phi(t_2)=\phi_2 \\ \phi(-T)=\phi_a \\ \phi(T)=\phi_b}} \mathcal{D}\phi(x)\ \phi(x_1)\,\phi(x_2)\ \exp\left[i\int_{-T}^{T} d^4x\ \mathcal{L}\right]$$

The integral $\int \mathcal{D}\phi$ is constrained at four times. Break up the integral into regions:

$$\left.\begin{array}{l} -T < t < t_1 \\ t_1 < t < t_2 \\ t_2 < t < T \end{array}\right\} \text{ if } t_1 < t_2$$

or,

$$\left.\begin{array}{l} -T < t < t_2 \\ t_2 < t < t_1 \\ t_1 < t < T \end{array}\right\} \text{ if } t_2 < t_1$$

The contribution to the path integral from each of these regions gives a transition amplitude. If $t_1 < t_2$,

$$= \int \mathcal{D}\phi_1(\vec{x})\, \mathcal{D}\phi_2(\vec{x})\ \phi_1(\vec{x}_1)\, \phi_2(\vec{x}_2)\ \langle \phi_b | e^{-iH(T-t_2)} | \phi_2 \rangle$$

$$\times \langle \phi_2 | e^{-iH(t_2-t_1)} | \phi_1 \rangle \langle \phi_1 | e^{-iH(t_1+T)} | \phi_a \rangle$$

If $t_2 < t_1$, exchange $t_1 \leftrightarrow t_2$.

The Schrödinger operator $\phi_S(\vec{x}_1)|\phi_1\rangle = \phi_1(\vec{x}_1)|\phi_1\rangle$.
The states satisfy the completeness relation, e.g.

$$\int \mathcal{D}\phi_1 \, |\phi_1\rangle\langle\phi_1| = 1.$$

We have, if $t_1 < t_2$,

$$\int \mathcal{D}\phi \, \phi(x_1)\, \phi(x_2) \exp\left[i \int_{-T}^{T} d^4x \, \mathcal{L}\right]$$

$$= \langle \phi_b | e^{-iH(T-t_2)} \phi_S(\vec{x}_2)\, e^{-iH(t_2-t_1)} \phi_S(\vec{x}_1)\, e^{-iH(t_1+T)} |\phi_a\rangle$$

The Heisenberg field is related to the Schrödinger field by

$$e^{iHt_2} \phi_S(\vec{x}_2) e^{-iHt_2} = \phi_H(t_2, \vec{x}_2)$$

If $t_2 < t_1$, reverse the order of $t_1, t_2$ above.
Accounting for the time-ordering we have

$$\boxed{\begin{aligned} &\int \mathcal{D}\phi \, \phi(x_1)\, \phi(x_2) \exp\left[i \int_{-T}^{T} d^4x \, \mathcal{L}\right] \\ &\quad = \langle \phi_b | e^{-iHT} \, T\left[\phi_H(x_1)\, \phi_H(x_2)\right] e^{-iHT} |\phi_a\rangle \end{aligned}}$$

If we take $T \to \infty(1-i\epsilon)$ then $e^{-iHT}|\phi_a\rangle$ projects onto the vacuum state.

$$e^{-iHT}|\phi_a\rangle = \sum_n e^{-iE_nT}|n\rangle\langle n|\phi_a\rangle \to \langle 0|\phi_a\rangle e^{-iE_0\infty(1-i\epsilon)}|0\rangle$$

All other energy eigenstates are suppressed by $e^{-E_n(\infty\epsilon)}$.

Hence,

$$\lim_{T\to\infty(1-i\epsilon)} \langle \phi_b | e^{-iHT}\, T[\phi_H(x_1)\,\phi_H(x_2)]\, e^{-iHT} | \phi_a \rangle$$

$$= \underbrace{\langle 0|\phi_a\rangle\langle\phi_b|0\rangle\, e^{-2iE_0\infty(1-i\epsilon)}}_{\int \mathcal{D}\phi\, \exp\left[i\int_{-T}^{T} d^4x\, \mathcal{L}\right]} \langle 0| T[\phi_H(x_1)\,\phi_H(x_2)]|0\rangle$$

Dividing by the extra phase, we get

$$\boxed{\langle 0| T[\phi_H(x_1)\,\phi_H(x_2)]|0\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi\, \phi(x_1)\,\phi(x_2)\, e^{i\int_{-T}^{T} d^4x\, \mathcal{L}}}{\int \mathcal{D}\phi\, e^{i\int_{-T}^{T} d^4x\, \mathcal{L}}}}$$

Similarly,

$$\langle 0| T[\phi_H(x_1)\cdots\phi_H(x_n)]|0\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi\, \phi(x_1)\cdots\phi(x_n)\, e^{i\int_{-T}^{T} d^4x\, \mathcal{L}}}{\int \mathcal{D}\phi\, e^{i\int_{-T}^{T} d^4x\, \mathcal{L}}}$$

<u>Functional Derivatives</u>

<u>Definition</u>: $\boxed{\dfrac{\delta}{\delta J(x)} J(y) = \delta^4(x-y)}$

<u>Example</u>: $\dfrac{\delta}{\delta J(x)} \int d^4y\, J(y)\,\phi(y) = \phi(x)$

<u>Example</u>: $\dfrac{\delta}{\delta J(x)} \exp\left[i\int d^4y\, J(y)\,\phi(y)\right] = i\,\phi(x)\, \exp\left[i\int d^4y\, J(y)\,\phi(y)\right]$

Example: $\frac{\delta}{\delta J(x)} \int d^4y\, \partial_\mu J(y)\, V^\mu(y)$

$$= \frac{\delta}{\delta J(x)} \int d^4y\, J(y)\left(-\partial_\mu V^\mu\right)$$

$$= -\partial_\mu V^\mu(x)$$

## Generating Functional for Correlation Functions

$$Z[J(x)] \equiv \int \mathcal{D}\phi \, \exp\left[i\int d^4x \left(\mathcal{L} + \underbrace{J(x)\,\phi(x)}_{\text{source term}}\right)\right]$$

$$\left(-i\frac{\delta}{\delta J(x_1)}\right)\left(-i\frac{\delta}{\delta J(x_2)}\right) Z[J]\Big|_{J=0} = \int \mathcal{D}\phi\, \phi(x_1)\,\phi(x_2)\, \exp\left[i\int d^4x\, \mathcal{L}\right]$$

Hence, $\langle 0| T[\phi(x_1)\,\phi(x_2)]|0\rangle = \frac{1}{Z[J=0]}\left(-i\frac{\delta}{\delta J(x_1)}\right)\left(-i\frac{\delta}{\delta J(x_2)}\right) Z[J]\Big|_{J=0}$

Similarly,

$$\langle 0| T[\phi(x_1)\,\phi(x_2)\cdots\phi(x_n)]|0\rangle$$

$$= \frac{1}{Z[J=0]}\left(-i\frac{\delta}{\delta J(x_1)}\right)\cdots\left(-i\frac{\delta}{\delta J(x_n)}\right) Z[J]\Big|_{J=0}.$$

Next we need tools for evaluation of functional integrals. We begin w/ the simplest class of functional integrals:

## Gaussian Functional Integrals

Consider the action for the free real scalar field:

$$S_{free} = \int d^4x \; \tfrac{1}{2}\left[\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2\right]$$

$$= \int d^4x \; \tfrac{1}{2} \phi \left[-\partial_\mu \partial^\mu - m^2\right] \phi \quad + \text{surface term}$$

$$\equiv \int d^4x \, d^4x' \; \tfrac{i}{2} \phi(x') K(x,x') \phi(x)$$

where

$$K(x,x') = i\, \delta^4(x-x') \left(\partial_\mu \partial^\mu + m^2\right)$$

We want to calculate $\lim\limits_{T \to \infty(1-i\epsilon)} \int \mathcal{D}\phi \; \phi(x_1) \cdots \phi(x_n) \, e^{i\int_{-T}^{T} d^4x \, \mathcal{L}}$

$$= \lim_{T \to \infty(1-i\epsilon)} \int \mathcal{D}\phi \; \phi(x_1) \cdots \phi(x_n) \exp\left[-\tfrac{1}{2}\int d^4x \, d^4x' \, \phi(x') K(x,x') \phi(x)\right]$$

We already know how to do Gaussian integrals over a finite number of variables, so we'll start there.

① Gaussian integral of a single variable $\phi$:

$$I_1 = \int_{-\infty}^{\infty} d\phi \, \exp\left[-\tfrac{K}{2}\phi^2 + h\phi\right] = \sqrt{\frac{2\pi}{K}} \; e^{h^2/2K}$$

Derivatives with respect to $h$ give integrals of Gaussians times powers of $\phi$:

$$\int_{-\infty}^{\infty} d\phi\, \phi \exp\left[-\frac{K}{2}\phi^2 + h\phi\right] = \sqrt{\frac{2\pi}{K}}\, e^{h^2/2K}\, \frac{h}{K}$$

$$\int_{-\infty}^{\infty} d\phi\, \phi^2 \exp\left[-\frac{K}{2}\phi^2 + h\phi\right] = \sqrt{\frac{2\pi}{K}}\, e^{h^2/2K} \left(\frac{1}{K} + \frac{h^2}{K^2}\right)$$

Define $$\langle \phi^n \rangle = \frac{\int_{-\infty}^{\infty} d\phi\, \phi^n \exp\left[-\frac{K}{2}\phi^2 + h\phi\right]}{\int_{-\infty}^{\infty} d\phi\, \exp\left[-\frac{K}{2}\phi^2 + h\phi\right]}$$

Define $\langle \phi^2 \rangle_c \equiv \langle \phi^2 \rangle - \langle \phi \rangle^2 = \frac{1}{K}$

② Gaussians with $N$ variables:

$$I_N = \int_{-\infty}^{\infty} \prod_{i=1}^{N} d\phi_i \exp\left[-\sum_{ij} \frac{1}{2} \phi_i K_{ij} \phi_j + \sum_i h_i \phi_i\right]$$

Assume the matrix $K_{ij}$ can be diagonalized.

$$\left.\begin{array}{ll} K_{ij} \text{ eigenvectors} & \hat{q}_i \\ \text{eigenvalues} & K_q \end{array}\right\} \quad (K\hat{q})_i = K_q \hat{q}_i \quad , \sum_i \hat{q}_i \cdot \hat{q}_i = 1.$$

$\hat{q} = 1, \dots, N$

$\sum_i \hat{q}^{(1)}_i \cdot \hat{q}^{(2)}_i = 0$ if $\hat{q}^{(1)} \neq \hat{q}^{(2)}$.

Write $\phi_i = \sum_q \hat{\phi}_q \hat{q}_i$ , $h_i = \sum_q \hat{h}_q \hat{q}_i$

$$I_N = \prod_{i=1}^{N} \int_{-\infty}^{\infty} d\hat{\phi}_i \exp\left[-\frac{K_i}{2}\hat{\phi}_i^2 + \hat{h}_i \hat{\phi}_i\right]$$

$$= \prod_{i=1}^{N} \sqrt{\frac{2\pi}{K_i}} \exp\left[\frac{\hat{h}_i K_i^{-1} \hat{h}_i}{2}\right]$$

In terms of the original parameters $K_{ij}$ and $h_i$,

$$\prod_{i=1}^{N} \tilde{K}_i = \det K$$

$$\sum_i \tilde{h}_i \tilde{K}_i^{-1} \tilde{h}_i = \sum_{ij} h_i K_{ij}^{-1} h_j$$

The inverse matrix $K_{ij}^{-1}$ satisfies $K K^{-1} = K^{-1} K = 1$.
We have,

$$\boxed{I_N = \sqrt{\frac{(2\pi)^N}{\det K}} \exp\left[\frac{1}{2} \sum_{ij} h_i K_{ij}^{-1} h_j\right]}.$$

Define
$$\langle \phi_{i_1} \cdots \phi_{i_n} \rangle = \frac{\int_{-\infty}^{\infty} \prod_{i=1}^{N} d\phi_i \; \phi_{i_1} \cdots \phi_{i_n} \exp\left[-\sum_{ij} \frac{1}{2} \phi_i K_{ij} \phi_j + \sum_i h_i \phi_i\right]}{\int_{-\infty}^{\infty} \prod_{i=1}^{N} d\phi_i \exp\left[-\sum_{ij} \frac{1}{2} \phi_i K_{ij} \phi_j + \sum_i h_i \phi_i\right]}$$

$$\langle \phi_i \rangle = \sum_j K_{ij}^{-1} h_j$$

$$\langle \phi_i \phi_j \rangle_c \equiv \langle \phi_i \phi_j \rangle - \langle \phi_i \rangle \langle \phi_j \rangle = K_{ij}^{-1}$$

③ Gaussian functional integrals.

The limit of an infinite # of variables.

$$\phi_i \to \phi(x)$$

$K_{ij} \to K(x, x')$ <u>Kernel</u> of the Gaussian integral

$$\boxed{\begin{aligned} &\int \mathcal{D}\phi \exp\left[-\int d^d x \, d^d x' \, \phi(x') \underbrace{K(x,x')}_{\textstyle\Sigma} \phi(x) + \int d^d x \, h(x) \phi(x)\right] \\ &\propto (\det K)^{-1/2} \exp\left[\int d^d x \, d^d x' \, h(x') \frac{K^{-1}(x,x')}{2} h(x)\right] \end{aligned}}$$

Formally, $(\det K)$ is the product of eigenvalues of the kernel, such that $K(x,x')\,\phi_i(x) = \delta^4(x-x')\,K_i\,\phi_i(x)$

The inverse kernel satisfies

$$\int d^dx'\, K(x,x')\,K^{-1}(x',x'') = \delta^d(x''-x)$$

An infinite constant $(2\pi)^{N/2}$ was factored out of the functional integral. Any constant disappears from the ratio of functional integrals yielding correlation functions. We have,

$$\boxed{\begin{aligned} &\langle\phi(x)\rangle = \int d^dx'\, K^{-1}(x,x')\,h(x') \\ &\langle\phi(x)\,\phi(x')\rangle_c \equiv \langle\phi(x)\,\phi(x')\rangle - \langle\phi(x)\rangle\langle\phi(x')\rangle = K^{-1}(x,x') \end{aligned}}$$

$\langle O\rangle \equiv \langle 0|T[O]|0\rangle$

Considering the case $K(x,x') = i\,\delta^4(x-x')\,(\partial^2+m^2)$,

$$\int \mathcal{D}\phi\, e^{iS_{free}} = (\text{const})\sqrt{\frac{1}{\det(\partial^2+m^2)}}$$

$$\boxed{\begin{aligned} \langle 0|\,T[\phi(x_1)\,\phi(x_2)]\,|0\rangle &= K^{-1}(x_1,x_2) \\ &= \int \frac{d^4k}{(2\pi)^4}\,\frac{i\,e^{-ik\cdot(x_1-x_2)}}{k^2-m^2+i\epsilon} \end{aligned}}$$

We have recovered the Feynman propagator. The $+i\epsilon$ follows from the clockwise rotation $t\to t(1-i\epsilon) \implies k^0 \to k^0(1+i\epsilon)$.