

We are now in a position to construct a non-Abelian gauge theory from knowledge of the gauge group and the representations under which the matter fields transform.

Example: Quantum Chromodynamics = The Strong Interaction

- SU(3) gauge theory
- Six flavors of quarks (u, d, c, s, t, b), each of which transforms in the fundamental representation of the SU(3) gauge group.

SU(3) \Rightarrow 8 gauge fields A_μ^a , $a=1, \dots, 8$
 (because $\dim \text{SU}(3) = 8$)

Fundamental rep = defining rep = 3-dimensional for SU(3)
 \Rightarrow 3 colors of quarks for each flavor.

Label quark fields \underline{q}_I^i , $I = u, d, c, s, t, b$ — flavor
 $i = 1, 2, 3$ — color

Generators of SU(3) \underline{T}^a , $a = 1, \dots, 8$

Gauge coupling \underline{g}_3

The Lagrangian for QCD is then!

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} + \sum_{I, i, j} \bar{q}_I^i (i \not{\partial} \delta_{ij} - g_s A^a T_{ij}^a) q_I^j$$

+ quark mass terms.

In the Standard Model quark (and lepton) masses are due to couplings to the Higgs fields, which requires one more ingredient.

Pseudoreality of $SU(2)$ Representations:

A representation of a group is called pseudoreal if there is a nonsingular matrix S such that

$$\boxed{S T^a S^{-1} = -T^{a*}}$$

In that case a representation is equivalent to its conjugate.

The $SO(N)$ groups have all real representations. Since $SU(2) \sim SO(3)$, the integer-spin reps of $SU(2)$ are real.

The half-odd-integer spin reps are called spinor reps. For them, there is a nontrivial S that relates the rep to its conjugate.

Consider the spin- $1/2$ representation:

$$T^1 = \frac{\sigma^1}{2}, \quad T^2 = \frac{\sigma^2}{2}, \quad T^3 = \frac{\sigma^3}{2}$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma^{1*} \Rightarrow \boxed{-T^{1*} = -T^1}$$

$$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -\sigma^{2*} \Rightarrow \boxed{-T^{2*} = +T^2}$$

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma^{3*} \Rightarrow \boxed{-T^{3*} = -T^3}$$

Let $S = -i\sigma^2$.

$$S T^1 S^{-1} = \sigma^2 \frac{\sigma^1}{2} \sigma^2 = -(\sigma^2)^2 \frac{\sigma^1}{2} = -\frac{\sigma^1}{2} = -T_1^*$$

$$S T^2 S^{-1} = \sigma^2 \frac{\sigma^2}{2} \sigma^2 = \frac{\sigma^2}{2} = -T_2^*$$

$$S T^3 S^{-1} = \sigma^2 \frac{\sigma^3}{2} \sigma^2 = -\frac{\sigma^3}{2} = -T_3^*$$

Hence, $S T^a S^{-1} = -T^a^*$ for $a=1,2,3$.

Suppose $\underline{\Phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ transforms as an $SU(2)$ doublet,

$$\underline{\Phi} \rightarrow \exp\left[i\theta^a \frac{\sigma^a}{2}\right] \underline{\Phi}$$

Then,

$$\underline{\Phi}^* \rightarrow \exp\left[-i\theta^a \frac{(\sigma^a)^T}{2}\right] \underline{\Phi}^*$$

(*) The combination $i\sigma^2 \underline{\Phi}^* = \begin{pmatrix} \phi_2^* \\ -\phi_1^* \end{pmatrix}$ transforms as

an $SU(2)$ doublet, as we will now check.

$$\begin{aligned} i\sigma^2 \underline{\Phi}^* &\rightarrow i\sigma^2 \exp\left[-i\theta^a \frac{(\sigma^a)^T}{2}\right] \underline{\Phi}^* \\ &= i\sigma^2 \sum_{n=0}^{\infty} \frac{1}{n!} \left[-i\theta^a \frac{(\sigma^a)^T}{2}\right]^n \underline{\Phi}^* \end{aligned}$$

Expand $\left[-i\theta^a \frac{(\sigma^a)^T}{2}\right]^n$ and insert $(-i\sigma^2)(i\sigma^2) = 1$ between factors of $(\sigma^a)^T$, and to the right of the last factor.

$$\begin{aligned} \text{Then, } i\sigma^2 \Phi^* &\rightarrow \sum_n \frac{1}{n!} \left[i\theta^a \frac{\sigma^a}{2} \right]^n i\sigma^2 \Phi^* \\ &= \exp \left[i\theta^a \frac{\sigma^a}{2} \right] (i\sigma^2 \Phi^*) \end{aligned}$$

This is what we wanted to prove:

Φ and $(i\sigma^2 \Phi^*)$ transform the same way.

The Higgs sector of the Standard Model takes advantage of this fact.

The Higgs fields $\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ are a pair of complex scalar fields that transform as an $SU(2)_W$ doublet under the weak interactions. — (This is a gauge invariance, not a global symmetry.)

The left-handed fermions (quarks or leptons) form $SU(2)_W$ doublets.

The right-handed fermions are $SU(2)_W$ singlets, i.e. they are invariant under $SU(2)_W$ transformations.

The Lagrangian $\mathcal{L}_{Yuk} = \lambda_d (\bar{\Psi}_L^1, \bar{\Psi}_L^2) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \Psi_R$ + h.c.

is $SU(2)_W$ -invariant. But so is

$$\mathcal{L}_{Yuk} = \lambda_u (\bar{\Psi}_L^1, \bar{\Psi}_L^2) i\sigma^2 \begin{pmatrix} \phi_1^* \\ \phi_2^* \end{pmatrix} \Psi_R' + \text{h.c.}$$

Both types of Yukawa couplings are required in the Standard Model for understanding fermion masses.

If the Higgs doublet has a potential $V(\phi_1, \phi_2)$ with a minimum at $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ v \end{pmatrix}$ with v real,

then expanding the Lagrangian about this minimum gives rise to terms of the form

$$\begin{aligned} \mathcal{L}_d &= \lambda_d (\bar{\Psi}_L^1, \bar{\Psi}_L^2) \begin{pmatrix} 0 \\ v \end{pmatrix} \Psi_R + \text{h.c.} \\ &= (\lambda_d v) \bar{\Psi}_L^2 \Psi_R + \text{h.c.} \end{aligned}$$

This looks like a mass term for the Dirac spinor whose left-handed component is Ψ_L^2 and whose right-handed component is Ψ_R , with mass $(\lambda_d v)$.

$$\begin{aligned} \mathcal{L}_u &= \lambda_u (\bar{\Psi}_L^1, \bar{\Psi}_L^2) \begin{pmatrix} v \\ 0 \end{pmatrix} \Psi_R' + \text{h.c.} \\ &= (\lambda_u v) \bar{\Psi}_L^1 \Psi_R' + \text{h.c.} \end{aligned}$$

This is a mass term for Ψ_L^1 and Ψ_R' .

If it weren't for pseudoreality of $SU(2)$ representations, we couldn't have written \mathcal{L}_u consistent w/ the weak $SU(2)_L$, and Ψ' would be massless.