Consequences of Global Symmetries

A global symmetry transformation is one which is the same everywhere in spacetime. This is to be contrasted with gauge invariances, for which an independent transformation is made at each point in spacetime.

Before constructing non-Abelian gauge theories we pause to examine some of the consequences of (approximate) non-Abelian global symmetries. As an example, we consider the approximate isospin symmetry of the Standard Model.

The fermions which participate in the strong interactions are called quarks. As a result of the strong interactions all bound states in the theory are massive, with a typical scale being the proton mass ~1 GeV. The lightest two quarks, the up and down quarks, have masses of a few MeV. To a good approximation, then, when considering the properties of hadrons the up and down can be considered approximately massless. In particular their mass difference can often be neglected.

Consider a free theory of two Dirac fermions of mass m:

$$\mathcal{L}_{free} = \sum_{i=1,2} \bar{\Psi}_i (i\partial\!\!\!/ - m) \Psi_i$$

The free Lagrangian is invariant under transformations of the form, $\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \to U\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$

$$(\bar{\psi}_1, \bar{\psi}_2) \to (\bar{\psi}_1, \bar{\psi}_2)U^+$$

where $U^+U = 1$. Hence, U is a 2×2 unitary matrix. The group of 2×2 unitary matrizes is called $U(2)$. By unitarity, $|\det U|^2 = 1$. Hence, $U = e^{i\theta}\tilde{U}$ for some phase θ, where $\det \tilde{U} = 1$. $\tilde{U}$ is also a unitary matrix, so $U(2) = U(1)\times SU(2)$. The $U(1)$ acts as an overall phase rotation of ψ_1 and ψ_2 (baryon #), w/ hypercharge
The $SU(2)$ transformations on the up and down quark are called isospin, short for isobaric spin because 1) it relates nuclei w/ the same baryon number and 2) $SU(2) \sim SO(3)$ w/ spinor representations.
(Adding the strong interactions to the free Lagrangian does not affect isospin.)
The generators of $SU(2)$ satisfy the $SU(2)$ algebra:

$$[I_i, I_j] = i\epsilon_{ijk} I_k .$$

Representations are $2I+1$ dimensional (think of rotations).
There is a basis which diagonalizes $\vec{I}^2$ and I_3:

$$I_3 = \begin{pmatrix} I & & & & & & \\ & I-1 & & & & & \\ & & I-2 & & & & \\ & & & \ddots\, 0 \,\ddots & & & \\ & & & & -(I-2) & & \\ & & & & & -(I-1) & \\ & & & & & & -I \end{pmatrix} \qquad \vec{I}^2 = I(I+1)$$

__Hadrons__ are bound states in QCD. Hadrons composed of light quarks should form representations of isospin. Some common multiplets are

$$\begin{array}{l} I_z=+\frac{1}{2} \\ I_z=-\frac{1}{2} \end{array}\begin{pmatrix} p \\ n \end{pmatrix}, \quad \begin{pmatrix} K^+ \\ K^0 \end{pmatrix}, \quad \begin{pmatrix} \bar{K}^0 \\ K^- \end{pmatrix} \qquad \vec{I}^{\,2} = \frac{1}{2}\left(\frac{1}{2}+1\right) = \frac{3}{4}$$

$$\begin{array}{l} I_z=+1 \\ I_z=0 \\ I_z=-1 \end{array}\begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix} \qquad \vec{I}^{\,2} = 1(1+1) = 2$$

$$I_z=0 \quad \Lambda, \eta \qquad \vec{I}^{\,2} = 0(0+1) = 0$$

The $+, 0, -$ superscripts are their electric charge, Q. Note that $(Q - I_3) \equiv \frac{Y}{2}$ is constant in each isospin multiplet. Y is called the __hypercharge__.

Consider $\pi N \to \pi N$ scattering. $N = p$ or n
$\pi = \pi^+, \pi^0$ or π^-

There are 6 possible initial states and 6 final states, but not all 36 scattering amplitudes are independent. Electric charge conservation rules out a bunch of them.

Allowed processes are:

Initial particles		Final Particles	# Amplitudes
$\pi^+ p$	$\longrightarrow$	$\pi^+ p$	1
$\pi^0 p$ $\pi^+ n$		$\pi^0 p$ $\pi^+ n$	3 ($\pi^0 p \to \pi^+ n$ is related to $\pi^+ n \to \pi^0 p$ by time reversal)
$\pi^- p$ $\pi^0 n$		$\pi^- p$ $\pi^0 n$	3
$\pi^- n$	$\longrightarrow$	$\pi^- n$	1

without using isospin we have cut down the number of independent amplitudes to 8.
With isospin we can cut them down to 2.

When we put a particle from an $I=1$ and a particle from an $I=\frac{1}{2}$ multiplet together we get states that belong to $I=\frac{3}{2}$ and $I=\frac{1}{2}$ multiplets, just like in the addition of ordinary spins: $1 \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{3}{2}$

π^+ and p are the highest weight members of their multiplets, so

$$|\pi^+ p\rangle = \left|I=\tfrac{3}{2},\ I_3=\tfrac{3}{2}\right\rangle$$

By applying isospin lowering operators $I_1 \pm I_2$ on the highest weight state, just like in the construction of Clebsch-Gordon coefficients in ordinary spin addition,

$$|\pi^+ n\rangle = \sqrt{\frac{1}{3}}\,|I=\tfrac{3}{2}, I_3=\tfrac{1}{2}\rangle + \sqrt{\frac{2}{3}}\,|I=\tfrac{1}{2}, I_3=\tfrac{1}{2}\rangle$$

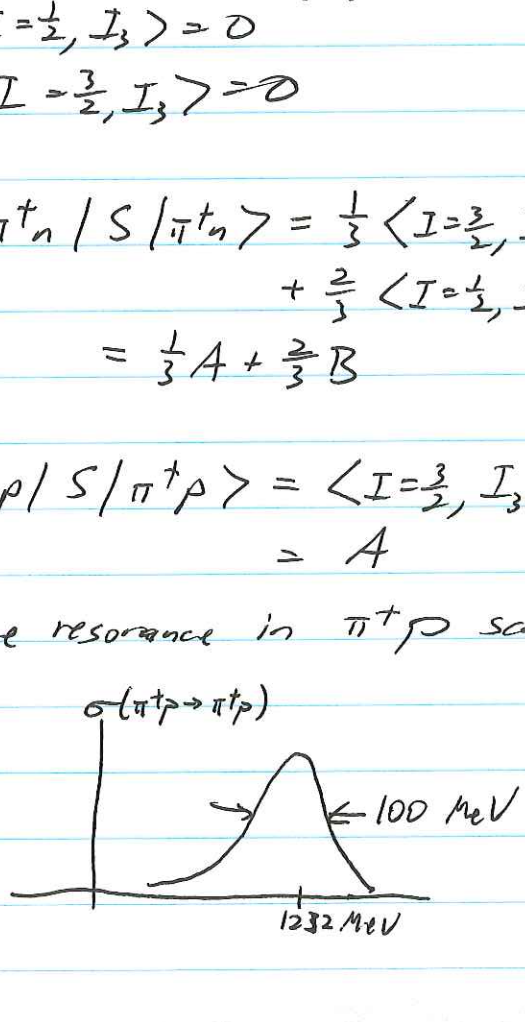
At fixed momenta and spins there are just two independent scattering amplitudes, A and B:

$$\langle I=\tfrac{3}{2}, I_3'|\,S\,|I=\tfrac{3}{2}, I_3\rangle = A\,\delta_{I_3 I_3'}$$
$$\langle I=\tfrac{1}{2}, I_3'|\,S\,|I=\tfrac{1}{2}, I_3\rangle = B\,\delta_{I_3 I_3'}$$
$$\langle I=\tfrac{3}{2}, I_3'|\,S\,|I=\tfrac{1}{2}, I_3\rangle = 0$$
$$\langle I=\tfrac{1}{2}, I_3'|\,S\,|I=\tfrac{3}{2}, I_3\rangle = 0$$

for example,
$$\begin{aligned}\langle \pi^+ n|\,S\,|\pi^+ n\rangle &= \tfrac{1}{3}\langle I=\tfrac{3}{2}, I_3=\tfrac{1}{2}|\,S\,|I=\tfrac{3}{2}, I_3=\tfrac{1}{2}\rangle \\ &\quad + \tfrac{2}{3}\langle I=\tfrac{1}{2}, I_3=\tfrac{1}{2}|\,S\,|I=\tfrac{1}{2}, I_3=\tfrac{1}{2}\rangle \\ &= \tfrac{1}{3}A + \tfrac{2}{3}B\end{aligned}$$

$$\begin{aligned}\langle \pi^+ p|\,S\,|\pi^+ p\rangle &= \langle I=\tfrac{3}{2}, I_3=\tfrac{3}{2}|\,S\,|I=\tfrac{3}{2}, I_3=\tfrac{3}{2}\rangle \\ &= A\end{aligned}$$

There is a large resonance in $\pi^+ p$ scattering called the $\Delta(1232)$:



As long as the $I=\frac{1}{2}$ scattering amplitude B is small near the resonance @ 1232 MeV, then

$$\langle \pi^+ n | S | \pi^+ n \rangle \approx \frac{1}{3} \langle \pi^+ p | S | \pi^+ p \rangle$$

Indeed, the same $\Delta(1232)$ resonance is seen in $\pi^+ n$ scattering with the right height.

Chiral Symmetry

In the approximation of massless up and down quarks, isospin is actually part of a larger symmetry called chiral symmetry. Recall that we can decompose the Lagrangian for a massless Dirac field into its left- and right-handed chiralities,

$$\psi_L \equiv \frac{1}{2}(1-\gamma_5)\psi$$
$$\psi_R \equiv \frac{1}{2}(1+\gamma_5)\psi$$

$$\mathcal{L}_{free} = \bar{\psi} i\partial\!\!\!/ \psi = \bar{\psi}_L i\partial\!\!\!/ \psi_L + \bar{\psi}_R i\partial\!\!\!/ \psi_R$$

with two approximately massless Dirac fields (the up and down quarks),

$$\mathcal{L}_{free} \approx \sum_{j=1,2} \left(\bar{\psi}_L^j i\partial\!\!\!/ \psi_L^j + \bar{\psi}_R^j i\partial\!\!\!/ \psi_R^j \right)$$

There is now an $SU(2) \times SU(2)$ symmetry that acts separately on the left-handed and right-handed fields. This is called chiral symmetry, and also has important consequences in QCD phenomenology.

Pion-Nucleon Interactions

The nucleon doublet $N = \begin{pmatrix} p \\ n \end{pmatrix}$ transforms under SU(2) isospin as

$$\boxed{N \to \exp\left[i\theta^a \frac{\sigma^a}{2}\right] N}$$

The pions form a triplet π^a, $a = 1,2,3$.

Define $\pi = \pi^a \frac{\sigma^a}{2}$.

Under an isospin transformation $\boxed{\pi \to e^{i\theta^a\frac{\sigma^a}{2}}\, \pi\, e^{-i\theta^a\frac{\sigma^a}{2}}}$

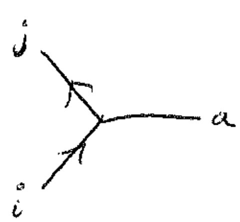
The Lagrangian $\mathcal{L} = \bar{N}(i\partial\!\!\!/ - m)N + \mathrm{Tr}(\partial_\mu \pi)(\partial^\mu \pi) - ig\bar{N}\gamma^5\pi N$.

$\mathcal{L}$ is SU(2) invariant (Exercise).

Feynman rules

$i \xrightarrow{\;p\;} j$ $\qquad \dfrac{i(\not{p}+m)}{p^2-m^2+i\epsilon}\,\delta_{ij}$

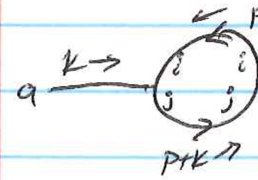
$a \overset{p\,\to}{\text{———}} b$ $\qquad \dfrac{i}{p^2+i\epsilon}\,\delta_{ab}$



$+\dfrac{g}{2}\gamma^5\,\sigma^a_{ji}$

Example: Pion Self-Energy

$$\int \langle 0 | T(\pi^a(x)\,\pi^b(0)) | 0 \rangle\, e^{+ip\cdot x}\, d^4x$$



$-i\,\Pi^{ab}(k^2)$

$$-i\,\Pi^{ab}(k^2) = \int \frac{d^4p}{(2\pi)^4}\; -\mathrm{Tr}\left[\left(+\frac{g}{2}\sigma^a_{ji}\right)\gamma^5 \frac{i}{\not p - m + i\epsilon} \times \left(+\frac{g}{2}\sigma^b_{ij}\right)\gamma^5 \frac{i}{\not p + \not k - m + i\epsilon}\right]$$

$$= -\frac{g^2}{4}\,\mathrm{Tr}\,\sigma^a\sigma^b \int \frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\left[\gamma^5 \frac{i}{\not p - m + i\epsilon}\gamma^5 \frac{i}{\not p + \not k - m + i\epsilon}\right]$$

$$= -\frac{g^2}{2}\,\delta^{ab} \int \frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\left[\gamma^5 \frac{i}{\not p - m + i\epsilon}\gamma^5 \frac{i}{\not p + \not k - m + i\epsilon}\right]$$

The group theoretic factor $\mathrm{Tr}\,\sigma^a\sigma^b$ factored out of the momentum integral.

Example: Nucleon Self-Energy

$$-i\,\Sigma_{ij}(k) = \int \frac{d^4p}{(2\pi)^4}\left(+\frac{g}{2}\sigma^a_{jk}\right)\gamma^5 \frac{i}{\not p + \not k - m + i\epsilon}\left(+\frac{g}{2}\sigma^a_{ki}\right)\gamma^5 \frac{i}{p^2 + i\epsilon}$$

$$= \frac{g^2}{4}(\sigma^a\sigma^a)_{ji} \int \frac{d^4p}{(2\pi)^4}\,\gamma^5 \frac{i}{\not p + \not k - m + i\epsilon}\gamma^5 \frac{i}{p^2 + i\epsilon}$$

$$\frac{1}{4}(\sigma^a\sigma^a)_{ji} = C_2(\text{fundamental rep})\,\delta_{ij} = \frac{3}{4}\delta_{ij}$$

quadratic Casimir $\qquad \frac{1}{2}\left(\frac{1}{2}+1\right)$ (cf. spin)

More about global symmetries

As a general rule, the *kinetic terms* of a field theory have at least as large a set of symmetries as the full theory.

Example: The kinetic terms of a theory of N Dirac fermions is

$$\mathcal{L}_{kin} = \sum_{i=1}^{N} \bar{\psi}_i \, i\partial\!\!\!/ \, \psi_i$$

$$= \sum_{i=1}^{N} \left(\bar{\psi}_L^i \, i\partial\!\!\!/ \, \psi_L^i + \bar{\psi}_R^i \, i\partial\!\!\!/ \, \psi_R^i \right)$$

The left- and right-handed components can be separately transformed by $U(N)$ symmetries which leave the Lagrangian invariant:
Collect the fermions in a column $\Psi_L = \begin{pmatrix} \psi_L^1 \\ \psi_L^2 \\ \vdots \\ \psi_L^N \end{pmatrix}$, etc.

$$\mathcal{L}_{kin} = \bar{\Psi}_L \, i\partial\!\!\!/ \, \Psi_L + \bar{\Psi}_R \, i\partial\!\!\!/ \, \Psi_R$$

If $\Psi_L \to U_L \Psi_L$ and $\Psi_R \to U_R \Psi_R$, where U_L and U_R are $N \times N$ unitary matrices, then

$$\mathcal{L}_{kin} \to i \bar{\Psi}_L U_L^\dagger \partial\!\!\!/ \, U_L \Psi_L + i \bar{\Psi}_R U_R^\dagger \partial\!\!\!/ \, U_R \Psi_R$$

$$= \mathcal{L}_{kin} \quad \text{because } U_L^\dagger U_L = U_R^\dagger U_R = \mathbb{1}.$$

(and U_L, U_R commute with the $\partial\!\!\!/$)

If the fermions all have the same mass m, then

$$\mathcal{L} = \mathcal{L}_{kin} + m\,\overline{\Psi}\,\Psi.$$
$$= \mathcal{L}_{kin} + m\left(\overline{\Psi}_L\,\Psi_R + \overline{\Psi}_R\,\Psi_L\right)$$

If $\Psi_L \to U_L\,\Psi_L$ and $\Psi_R \to U_R\,\Psi_R$, then

$$\mathcal{L} \to \mathcal{L}_{kin} + m\left(\overline{\Psi}_L\,U_L^{+}U_R\,\Psi_R + \overline{\Psi}_R\,U_R^{+}U_L\,\Psi_L\right)$$

The theory of free massive fermions has less symmetry than the massless theory.
$\mathcal{L}$ is only invariant under transformations such that $U_L = U_R$.

* Hence, the mass terms explicitly break the $U(N)\times U(N)$ chiral symmetry to a $U(N)$ subgroup.

If the fermions have different masses m_i, then

$$\mathcal{L} = \mathcal{L}_{kin} + \sum_i m_i\left(\overline{\Psi}_L{}^i\,\psi_R{}^i + \overline{\Psi}_R{}^i\,\psi_L{}^i\right)$$

$\mathcal{L}$ is invariant under a $U(1)^N$ subgroup of the $U(N)\times U(N)$ chiral symmetry

$$\psi_L{}^i \to U_i\,\psi_L{}^i\,, \quad \psi_R{}^i \to U_i\,\psi_R{}^i$$

$$U_j = e^{i\theta_j}$$