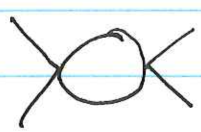
Minimal Subtraction

The physical renormalization prescription we have been using is not unique. There are alternatives which leave the result of loop calculations looking simpler, but the cost is that one has to work to obtain physical results.

Minimal Subtraction (MS) is a renormalization prescription that goes together with dimensional regularization. The scheme is to choose counterterms that cancel the poles as $d \to 4$, but that's it! You don't worry about physical masses and physical couplings; you just throw away the poles.

In the MS scheme we would have (Exercise)

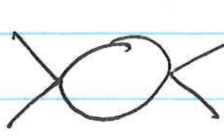
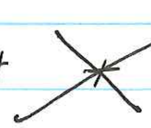
 + 

(counterterm)

$$\sim \frac{\lambda^2 \pi^2}{(2\pi)^4}\left[-\gamma_E + \log\left(\frac{\mu^2}{\pi q^2}\right)\right]$$

q = function of momenta, masses.

There is also a Modified Minimal Subtraction ($\overline{MS}$) scheme in which the unphysical finite constants are also dropped:

 + 

$$\sim \frac{\lambda^2 \pi^2}{(2\pi)^4} \log\left(\frac{\mu^2}{q^2}\right)$$

General Loop Diagrams

We already have the tools to express any one-loop Feynman diagram as an integral over a finite number of Feynman parameters. We can generalize those techniques to multi-loop diagrams.

Consider a diagram w/ L loops and I internal lines, eg.

$L = 2 \rightarrow \int d^4k_1\, d^4k_2$
$I = 5 \rightarrow 5$ propagators

The denominators from the propagators can be combined by a Feynman parametrization. Then the diagram is of the form,

$$\int_0^1 dx_1 \cdots dx_I \, \delta(1 - \Sigma x_i) \int \frac{d^4k_1 \cdots d^4k_L}{(2\pi)^{4L} \, D^I} N$$

where D is of the form,

$$D = \sum_{i,j=1}^{L} A_{ij} \, k_i \cdot k_j + \sum_{i=1}^{L} B_i \cdot k_i + C$$

A_{ij}: $L \times L$ matrix fn. of Feynman parameters

B_i: vector w/ L 4-vector components

C: #, linearly dependent on Feynman parameters, external momenta^2, masses^2. $Im(C) = i\epsilon$

N = function of momenta, γ-matrices.

Next, complete the squares: Eliminate the terms linear in k_i by shifting the integrations over k_i:

$$k_i' = k_i + \frac{1}{2}\sum_j (A^{-1})_{ij} B_j \quad , \quad d^4k_i = d^4k_i'$$

$$D = \sum_{ij=1}^{L} A_{ij}\, k_i' \cdot k_j' + C' \quad , \quad C' = C - \frac{1}{4}\sum_{ij} B_i A^{-1}_{ij} B_j$$

Next diagonalize A_{ij} w/ an orthogonal transformation on the 4-vectors k_i':

$$k_i' = O_{ij} k_j'' \quad , \quad \det O = 1$$

$$\prod_{i=1}^{L} d^4k_i' = \prod_{i=1}^{L} d^4k_i''$$

$$\left.\begin{aligned} D &= \sum_{\substack{ij=1\\ kl}}^{L} A_{ij}\, O_{ik}\, k_k'' \cdot k_l''\, O_{jl} + C' \\ &= \sum_{kl=1}^{L} (O^T A O)_{kl}\, k_k'' \cdot k_l'' + C' \\ &= \sum_{i=1}^{L} a_i\, k_i'' \cdot k_i'' + C' \end{aligned}\right\}$$

Note: In the end it won't matter what O_{ij} or a_i are. Only $(\det A)$ will appear in the final result.

where O is chosen s.t. $(O^T A O)_{kl} = a_l \delta_{kl}$

Finally, redefine k_i'' to eliminate the a_i:

$$k_i'' = \frac{1}{\sqrt{a_i}} k_i'''$$

$$\prod_{i=1}^{L} d^4 k_i''' = \prod_{i=1}^{L} (\sqrt{a_i})^4 d^4 k_i''$$

$$= (\det A)^2 \prod_{i=1}^{L} d^4 k_i''$$

$$D = \sum_{i=1}^{L} k_i''' \cdot k_i''' + C'$$

The integral to be done is now of the form

$$\int_0^1 dx_1 \cdots dx_I \, \delta(1-\sum x_i) \int (\det A)^{-2} \frac{d^4 k_1''' \cdots d^4 k_L'''}{(2\pi)^{4L} D^I} N(k_i''')$$

The terms in N that are odd in k_i''' integrate to 0. Symmetry arguments can often simplify the terms in N even in k_i'''.

To make the integrals spherically symmetric (rather than Lorentz symmetric), Wick rotate each of the $k_i^{0\,'''}$. The integral becomes

$$\int_0^1 dx_1 \cdots dx_I \, \delta(1-\sum x_i) \int \frac{d^4 k_{1E} \cdots d^4 k_{LE}}{(2\pi)^{4L} D^I} N(k_{iE}) (\det A)^{-2} i^L$$

$$d^4 k_i''' = i \, d^4 k_{iE} \, , \quad k_{i4} = -i k_i^{0\,'''}$$

$$D = -\sum_{i=1}^{L} k_{iE}^2 + C'$$

Spherical symmetry allows you to perform the 4L momentum integrals (if you are lucky and clever).

The result is an integral over I Feynman parameters, or I-1 parameters if you do the trivial δ-function integral over one of the I parameters.