The Dirac Form Factor and Infrared Divergences

The renormalized electron vertex function has the form,
$-i\tilde{\Gamma}^\mu(p,p') = -ie\gamma^\mu F_1(q^2) + \frac{\sigma^{\mu\nu}}{2m} q_\nu \, e F_2(q^2)$.

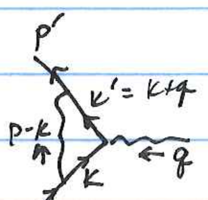
As you will show for homework, at one-loop, the Dirac form factor $F_1(q^2)$ may be written,

$$\boxed{\begin{aligned} F_1(q^2) = 1 + \frac{e^2}{8\pi^2}\int_0^1 dx \int_0^1 dy \int_0^1 dz \; \delta(x+y+z-1) \\ \Bigg[\log\frac{m^2(1-z)^2}{m^2(1-z)^2 - q^2xy} + \frac{m^2(1-4z+z^2) + q^2(1-x)(1-y)}{m^2(1-z)^2 - q^2xy} \\ - \frac{m^2(1-4z+z^2)}{m^2(1-z)^2}\Bigg] + O(e^4). \end{aligned}}$$

The log term includes a counterterm that cancelled a UV divergence and helped set $F_1(0) = 1$.

The Feynman parameter integrals of the remaining terms diverge from the region $x \approx y \approx 0$, $z \approx 1$.
Let's first trace where this problem came from.

The denominator in the Feynman integral before doing the momentum integration was:



$$D = \left[x(k^2 - m^2) + y(k'^2 - m^2) + z(k-p)^2 + (x+y+z)i\epsilon\right]^3$$

March 2011
1/8

Infrared Divergences

Consider the renormalized electron vertex function,

$$-i\tilde{\Gamma}^{\mu}(p,p') = -ie\gamma^{\mu}F_1(q^2) + \frac{\sigma^{\mu\nu}}{2m}q_\nu e F_2(q^2)$$

From last time (cf. your homework),

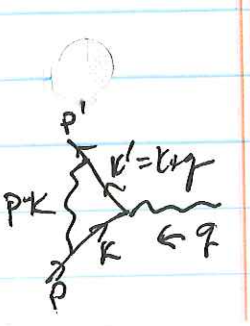
$$F_1(q^2) = 1 + \frac{e^2}{8\pi^2}\int_0^1 dx\int_0^1 dy\int_0^1 dz\ \delta(x+y+z-1)$$

$$\left[\log\frac{m^2(1-z)^2}{m^2(1-z)^2-q^2xy} + \frac{m^2(1-4z+z^2)+q^2(1-x)(1-y)}{m^2(1-z)^2-q^2xy}\right.$$

$$\left. - \frac{m^2(1-4z+z^2)}{m^2(1-z)^2}\right] + O(e^4)$$

The log term in the integral came from a UV divergence which was cancelled by a counterterm which set $F_1(0)=1$.

The Feynman parameter integrals of the remaining terms diverge from the region $x \approx y \approx 0$, $z \approx 1$. Let's first trace where the problem came from. The denominator in the Feynman integral before doing the loop integration was



$$D^3 = \left[x(k^2-m^2)+y(k'^2-m^2)+z(k-p)^2+(x+y+z)i\epsilon\right]^3$$

When $x \approx y \approx 0$, $z \approx 1$, the denominator D vanishes when the loop momentum k satisfies $k-p \approx 0$. This is when the internal photon line has vanishing momentum, so it is refered to as an <u>infrared divergence</u>.

To regulate the infrared divergence we can introduce a <u>photon mass</u> parameter μ, which we will eventually take to zero. The factor $(k-p)^2 z$ in D comes from the photon propagator so we replace it by $[(k-p)^2 - \mu^2] z$. The net result is to add the term $\mu^2 z$ to the denominators in $F_1(q^2)$.

Considering the terms which diverge when $x \approx y \approx 0$, $z \approx 1$,

$$F_1(q^2) \sim 1 + \frac{e^2}{8\pi^2} \int_0^1 dz \int_0^{1-z} dy \left[\frac{-2m^2 + q^2}{m^2(1-z)^2 - q^2 y(1-z-y) + \mu^2} - \underbrace{\frac{-2m^2}{m^2(1-z)^2 + \mu^2}}_{\delta F_1(0)} \right]$$

Change variables $\xi = \frac{y}{1-z}$, $w = 1-z$; $dz\, dy = \frac{1}{2} d\xi\, d(w^2)$

$$F_1(q^2) = 1 + \frac{e^2}{8\pi^2} \int_0^1 d\xi \cdot \frac{1}{2} \int_0^1 d(w^2) \left[\frac{-2m^2 + q^2}{w^2(m^2 - q^2 \xi(1-\xi)) + \mu^2} - \frac{-2m^2}{m^2 w^2 + \mu^2} \right]$$

$$F_1(q^2) \simeq 1 + \frac{e^2}{8\pi^2}\int_0^1 d\xi \cdot \frac{1}{2}\left[\frac{-2m^2+q^2}{m^2-q^2\xi(1-\xi)}\log\left(\frac{m^2-q^2\xi(1-\xi)}{\mu^2}\right) + 2\log\frac{m^2}{\mu^2}\right]$$

If $q^2 < 4m^2$ the divergence as $\mu \to 0$ is insensitive to the numerators in the logs.

Details unimportant as $\mu \to 0$.

$$\boxed{F_1(q^2) \sim 1 - \frac{e^2}{8\pi^2}\left[\int_0^1 \frac{m^2-q^2/2}{m^2-q^2\xi(1-\xi)}\,d\xi - 1\right]\log\left(\frac{-q^2 \text{ or } m^2}{\mu^2}\right)}$$

We have succeeded in isolating the IR divergence. The effect of $F_1(q^2)$ on the cross section for scattering off a potential is just the replacement $e \to e\,F_1(q^2)$.

$$\boxed{\frac{d\sigma}{d\Omega} \simeq \left(\frac{d\sigma}{d\Omega}\right)_{\text{tree level}}\left[1 - \frac{e^2}{4\pi^2}\left(\int_0^1 \frac{m^2-q^2/2}{m^2-q^2\xi(1-\xi)}\,d\xi - 1\right)\log\left(\frac{-q^2 \text{ or } m^2}{\mu^2}\right)\right]}$$

The physical reason for the IR divergence is that any experiment can only measure emitted photons down to some energy E_{min}. When using a cross section to count events, you should add the cross sections for (that event + photon w/ $E < E_{min}$), + (that event + 2 photons w/ $E < E_{min}$), etc. This was suggested by <u>Bloch and Nordsieck</u> in 1937 (before relativistic perturbation theory existed).

Note that you have to add the cross sections, not the amplitudes, for all of these events.

We will study how the emission of low energy photons cancels the IR divergence in $F_1(q^2)$ in the limit $-q^2 \gg m^2$.

Then $$\boxed{F_1(q^2)} \simeq 1 - \frac{e^2}{8\pi^2}\int_0^1 d\xi \, \frac{-q^2/2}{-q^2\xi(1-\xi)+m^2} \log\left(\frac{-q^2}{\mu^2}\right)$$

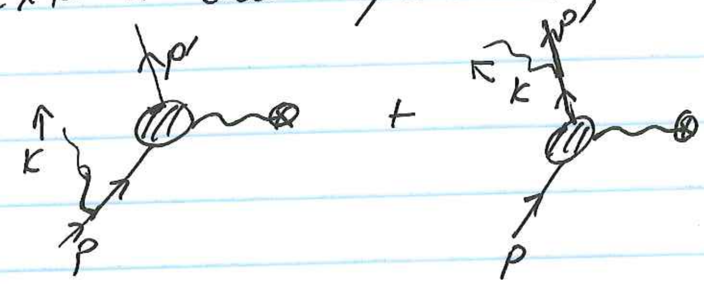
$$\boxed{\simeq 1 - \frac{e^2}{8\pi^2} \underbrace{\log\left(\frac{-q^2}{m^2}\right)\log\left(\frac{-q^2}{\mu^2}\right)}_{\text{"Sudakov double log"}}}$$

We will find the same IR-divergent double log will appear in the cross section for emission of soft photons, and will cancel the IR divergence from $F_1(q^2)$.

5/18

<u>Bremsstrahlung</u> – the radiation of photons by a charged particle when it interacts.

Consider the interaction of an electron w/ an external electromagnetic field



Call M_0 the part of the amplitude coming from the interaction w/ the external field. Then the amplitude for the Bremsstrahlung process is,

$$i\mathcal{M} = \bar{u}(p')\Big[M_0(p', p-k)\,\frac{i(\not p - \not k + m)}{(p-k)^2 - m^2 + i\epsilon}\, e\gamma^\mu \epsilon_\mu^*(k)$$

$$+ e\gamma^\mu \epsilon_\mu^*(k)\,\frac{i(\not p' + \not k + m)}{(p'+k)^2 - m^2 + i\epsilon}\, M_0(p'+k, p)\Big]u(p)$$

If the emitted photon is soft, then $M_0(p', p-k) \approx M_0(p', p)$ and $M_0(p'+k, p) \approx M_0(p', p)$.

The numerators are both simplified when $k \approx 0$:

$$(\not p + m)\gamma^\mu u(p) = \Big[2p^\mu + \gamma^\mu \underbrace{(-\not p + m)}_{0}\Big]u(p)$$
$$= 2p^\mu u(p)$$

$$\bar{u}(p')\gamma^\mu(\not{p}'+m) = \bar{u}(p')\left[2p'^\mu + \underbrace{(-\not{p}'+m)}_{0}\gamma^\mu\right]$$

$$= \bar{u}(p')\cdot 2p'^\mu$$

The denominators become: $(p-k)^2 - m^2 \approx -2p\cdot k$
$(p'+k)^2 - m^2 \approx +2p'\cdot k$

So, $$iM \approx \underbrace{\bar{u}(p')M_0(p',p)u(p)}_{\text{Amplitude for elastic scattering w/o bremsstrahlung.}}\left[e\left(\frac{p'\cdot\epsilon^*}{p'\cdot k} - \frac{p\cdot\epsilon^*}{p\cdot k}\right)\right]$$

Summing over polarization states of the photon, the differential cross section gets a factor,

$$d\sigma(p\to p'+\gamma) = d\sigma(p\to p')\int\frac{d^3k}{(2\pi)^3 2\underbrace{\omega_k}_{\omega_k = k}}\sum_{\lambda=1,2} e^2\left|\frac{p'\cdot\epsilon^{(\lambda)}}{p'\cdot k} - \frac{p\cdot\epsilon^{(\lambda)}}{p\cdot k}\right|^2$$

For the photon polarization sum we use $\sum_\lambda \epsilon_\mu^{(\lambda)}\epsilon_\nu^{(\lambda)*} \to -g_{\mu\nu}$

$$d\sigma(p\to p'+\gamma) = d\sigma(p\to p')\int\frac{d^3k}{(2\pi)^3 2\omega_k} e^2(-g_{\mu\nu})\left(\frac{p'^\mu}{p'\cdot k} - \frac{p^\mu}{p\cdot k}\right)\left(\frac{p'^\nu}{p'\cdot k} - \frac{p^\nu}{p\cdot k}\right)$$

$$= d\sigma(p\to p')\int\frac{d^3k}{(2\pi)^3 2\omega_k} e^2\left(\frac{2p\cdot p'}{(k\cdot p')(k\cdot p)} - \frac{m^2}{(p'\cdot k)^2} - \frac{m^2}{(p\cdot k)^2}\right)$$

The angular integral can be done by choosing a nice frame, like $p^0 = p'^0$. The resulting integral is of the form,

$$d\sigma(p \to p' + \gamma) = d\sigma(p \to p') \int dK \, \frac{e^2}{4\pi^2} \frac{1}{K} \underbrace{I(\vec{p}, \vec{p}\,')}_{\text{Independent of } K}$$

The K integral gives a log from the region where the emitted photon has $K \approx 0$.
If the photon had a small mass we could approximate the integral by replacing the lower bound by μ.

Since we are assuming $K \ll |\vec{p}\,' - \vec{p}| = |\vec{q}|$, we cutoff the integral at some $K \sim |\vec{q}|$.

Hence, $d\sigma(p \to p' + \gamma) \simeq d\sigma(p \to p') \frac{e^2}{8\pi^2} \log\left(\frac{-q^2}{\mu^2}\right) I(\vec{p}, \vec{p}\,')$

The factor $I(\vec{p}, \vec{p}\,')$ can be evaluated as $-q^2 \to \infty$, and gives $I(\vec{p}, \vec{p}\,') \to 2 \log\left(-\frac{q^2}{m^2}\right)$.

So, $$\boxed{d\sigma(p \to p' + \gamma) \approx d\sigma(p \to p') \frac{e^2}{4\pi^2} \log\left(\frac{-q^2}{\mu^2}\right) \log\left(\frac{-q^2}{m^2}\right)}$$

$-q^2 \gg m^2$.

We have recovered the double log that appeared in the IR divergent contribution to the cross section w/o bremsstrahlung from $F_1(q^2)$, but w/ the opposite sign.

As $\mu \to 0$ the divergent parts exactly cancel.

The probability of a scattering event with or without an undetected photon w/ energy $< E_{min}$ is proportional to,

$$\left.\frac{d\sigma}{d\Omega}\right|_{measured} = \frac{d\sigma}{d\Omega}(p \to p') + \frac{d\sigma}{d\Omega}(p \to p' + \gamma(k < E_{min}))$$

$$\to \left.\frac{d\sigma}{d\Omega}(p \to p')\right|_{\substack{tree\\level}} \left[1 + \frac{e^2}{8\pi^2} I(\vec{p}, \vec{p}\,') \log \frac{E_{min}^2}{\underbrace{f(m^2 \text{ and } -q^2)}_{\text{Requires more detailed calculation}}}\right]$$

$$\xrightarrow{-q^2 \gg m^2} \left.\frac{d\sigma}{d\Omega}(p \to p')\right|_{\substack{tree\\level}} \left[1 - \frac{e^2}{4\pi^2} \log\left(\frac{-q^2}{m^2}\right) \log\left(\frac{-q^2}{E_{min}^2}\right) + \mathcal{O}(e^4)\right]$$

The final result depends on the experimental conditions through E_{min}, but not on the fake photon mass μ.

By carefully summing over the cross sections for bremsstrahlung of arbitrary numbers of photons, one obtains for $-q^2 \gg m^2$,

$$\boxed{\left.\frac{d\sigma}{d\Omega}\right|_{measured} = \left.\frac{d\sigma}{d\Omega}\right|_{\substack{tree\\level}} \underbrace{\left[\exp\left(-\frac{e^2}{8\pi^2} \log\left(\frac{-q^2}{m^2}\right) \log\left(\frac{-q^2}{E_{min}^2}\right)\right)\right]^2}_{\text{"Sudakov form factor"}}}$$