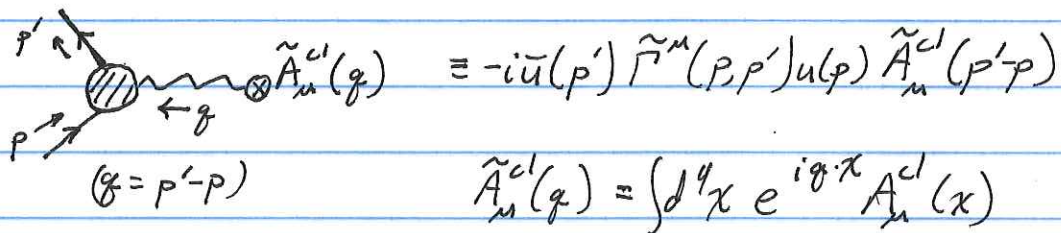


Electron Vertex Function

Consider scattering of an electron off of a background electromagnetic field. The corresponding Feynman diagrams have the form:



$$\begin{aligned}
 & \text{Diagram: } \text{Electron } (p) \rightarrow \text{Vertex} \rightarrow \text{Electron } (p') + \text{Background Field } (q) \\
 & \text{Equation: } -i \bar{u}(p') \Gamma^\mu(p, p') u(p) \tilde{A}_m^{cl}(q) \\
 & \text{Definition: } \tilde{A}_m^{cl}(q) = \int d^4x e^{iq \cdot x} A_m^{cl}(x)
 \end{aligned}$$

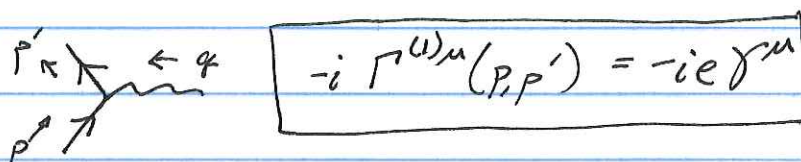
You can think of the background electromagnetic field as due to an external current j_μ .

In Feynman gauge, $\square A_m^{cl} = e j_\mu$, i.e.

$$\tilde{A}_m^{cl}(q) = -\frac{e}{q^2} \tilde{j}_m(q)$$

The 4x4 matrix function of p, p' $\Gamma^\mu(p, p')$ is called the electron vertex function.

At tree level,



$$-i \Gamma^{(0)\mu}(p, p') = -ie \gamma^\mu$$

Higher order corrections to the vertex function include

$$-i \tilde{\Gamma}^{\mu}(p, p') = \text{tree} + \text{self-energy} + \text{vertex correction} + \dots$$

+ CT

The counterterm comes from $\mathcal{L}_{CT} = -D \bar{\Psi} e \not{A} \Psi$,

$$\boxed{-i \tilde{\Gamma}^{\mu}(p, p') = -i D e \gamma^{\mu}}$$

Higher order corrections will not all be proportional to γ^{μ} . Lorentz invariance and gauge invariance restrict the possible matrix structures appearing in $\tilde{\Gamma}^{\mu}$. We can decompose $\tilde{\Gamma}^{\mu}$ as:

$$\tilde{\Gamma}^{\mu}(p, p') = A \gamma^{\mu} + B (p'^{\mu} + p^{\mu}) + C (p'^{\mu} - p^{\mu})$$

When $\tilde{\Gamma}^{\mu}(p, p')$ appears in an S-matrix element it is sandwiched between spinor wavefunctions $\bar{u}(p')$ and $u(p)$. The on-shell momenta and wavefunctions satisfy $p^2 = p'^2 = m^2$, $\not{p} u(p) = m u(p)$
 $\bar{u}(p') \not{p}' = m \bar{u}(p')$

We can think of A, B, C as functions of $(p' - p)^2$, as any other Lorentz invariants are redundant, i.e.

$$A = A(q^2), \quad B = B(q^2), \quad C = C(q^2), \quad q^{\mu} = p'^{\mu} - p^{\mu}.$$

(Note that we didn't include matrix structures including γ_5 because they would violate parity.)

Another Ward Identity due to gauge invariance

implies that inside S-matrix elements, $q_\mu \tilde{\Gamma}^\mu = 0$

$$0 = q_\mu \bar{u}(p') \tilde{\Gamma}^\mu(p, p') u(p)$$

$$= A(q^2) \bar{u}(p') (\not{p}' - \not{p}) u(p) + B(q^2) \bar{u}(p') u(p) (p'^m - p^m) (p'_m + p_m) + C(q^2) q^2$$

Since $\not{p}' u(p) = \bar{u}(p') \not{p}' u(p)$, the term $\propto A(q^2)$ vanishes.

Since $p^2 = p'^2 = m^2$, the term proportional to $B(q^2)$ vanishes.

If the photon is not on-shell, $q^2 \neq 0$. Hence, the Ward Identity implies $C(q^2) = 0$. Hence $\tilde{\Gamma}^\mu$ takes the form,

$$\tilde{\Gamma}^\mu(p, p') = A(q^2) \gamma^\mu + B(q^2) (p'^\mu + p^\mu)$$

It is customary to rewrite the second term using an identity for solutions to the Dirac equation $u(p), u(p')$:

$$\bar{u}(p') \gamma^\mu u(p) = \frac{1}{2m} \bar{u}(p') (p'^\mu + p^\mu + i \sigma^{\mu\nu} q_\nu) u(p)$$

$$\text{where } \sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$$

Gordon identity

The Gordon identity follows from

$$\begin{aligned}\gamma^\mu \not{p} &= \frac{1}{2} \{\gamma^\mu, \not{p}\} + \frac{1}{2} [\gamma^\mu, \not{p}] \\ &= p^\mu - i\sigma^{\mu\nu} p_\nu\end{aligned}$$

and $\not{p}' \gamma^\mu = p'^\mu + i\sigma^{\mu\nu} p'_\nu$. Then,

$$\begin{aligned}\bar{u}(p') \gamma^\mu u(p) &= \frac{1}{2m} \bar{u}' (\not{p}' \gamma^\mu + \gamma^\mu \not{p}) u \\ &= \frac{1}{2m} \bar{u}(p') (p'^\mu + p^\mu + i\sigma^{\mu\nu} q_\nu) u(p) \text{ as claimed.}\end{aligned}$$

The part $\propto p'^\mu + p^\mu$ has the form of the term in $\bar{u} \tilde{\Gamma}^\mu u$ multiplying $B(q^2)$. Hence, instead of using γ^μ and $(p'^\mu + p^\mu)$ in the decomposition of $\tilde{\Gamma}^\mu$, we can use γ^μ and $i\sigma^{\mu\nu} q_\nu$:

$$\tilde{\Gamma}^\mu(p, p') = e \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2m} e F_2(q^2)$$

Our goal is to interpret the form factors $F_1(q^2)$ and $F_2(q^2)$ and calculate them at 1-loop.

$F_1(q^2) \equiv$ Dirac Form Factor

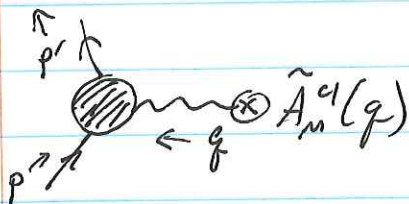
$F_2(q^2) \equiv$ Pauli Form Factor.

At tree level, $F_1(q^2) = 1$, $F_2(q^2) = 0$.

To understand the physical meaning of the form factors we consider scattering of a nonrelativistic electron from a background electromagnetic field.

First, consider a background electrostatic potential $\phi(\vec{x})$, s.t. $A_m^{cl}(x) = (\phi(\vec{x}), \vec{0})$.

$$\tilde{A}_m^{cl}(q) = \int d^4x e^{i\delta \cdot x} A_m^{cl}(x) = (2\pi \delta(q^0) \tilde{\phi}(\vec{q}), \vec{0})$$



$$iM = -i \bar{u}(p') \tilde{\Gamma}^m(p, p') u(p) \tilde{A}_m^{cl}(p' - p) \\ = -i \bar{u}(p') \tilde{\Gamma}^0(p, p') u(p) \tilde{\phi}(\vec{p}' - \vec{p})$$

(with the $2\pi\delta(q^0)$ factored out)

If the electrostatic field varies slowly over a large region, then $\tilde{\phi}(\vec{q})$ will be concentrated at $\vec{q} \approx \vec{0}$.

$$\text{In that limit, } \tilde{\Gamma}^0(p, p') = e \delta^0 F_1(q^2) + \frac{i\sigma^{0j}}{2m} e g_j F_2(q^2) \\ \rightarrow e \delta^0 F_1(q^2)$$

In the nonrelativistic limit $\bar{u}(p') \delta^0 u(p) = 2m \xi'^{\dagger} \xi$.
The scattering amplitude becomes,

$$iM \approx -ie F_1(0) \tilde{\phi}(\vec{q}) \cdot 2m \xi'^{\dagger} \xi$$

This agrees with the Born approximation for scattering off of a potential $V(\vec{x}) = e F_1(0) \phi(\vec{x})$.

Hence, $e F_1(0)$ is the electric charge of the electron.

Since $F_1(0) = 1$ at tree level, radiative corrections to $F_1(q^2)$ should vanish at $q^2 = 0$. \rightarrow Renormalization Condition

Now consider scattering off of a magnetic field specified by a static vector potential $A_m^{cl}(x) = (0, \vec{A}^{cl}(\vec{x}))$.

Then the scattering amplitude is,

$$iM = \sum_j i e \left[\bar{u}(p') \left(\gamma^j F_1(q^2) + i \frac{\sigma^{j\nu} q_\nu}{2m} F_2(q^2) \right) u(p) \right] \tilde{A}_d^j(\vec{q})$$

Again we assume that $\tilde{A}_d^j(\vec{q})$ is peaked near $\vec{q} = \vec{0}$.

Consider the spinors in the Weyl basis, expanded in \vec{p} and \vec{p}' :

$$u(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi \\ \sqrt{p \cdot \bar{\sigma}} \xi \end{pmatrix} \approx \sqrt{m} \begin{pmatrix} (1 - \frac{\vec{p} \cdot \vec{\sigma}}{2m}) \xi \\ (1 + \frac{\vec{p} \cdot \vec{\sigma}}{2m}) \xi \end{pmatrix}$$

Consider the $F_1(q^2)$ term in iM :

$$iM \supset i e \bar{u}(p') \gamma^j F_1(q^2) u(p) \tilde{A}_d^j(\vec{q})$$

Exercise: $\approx i e \cdot 2m \xi'^{\dagger} \left(\frac{\vec{p}' \cdot \vec{\sigma}}{2m} \sigma^j + \sigma^j \frac{\vec{p} \cdot \vec{\sigma}}{2m} \right) \xi F_1(0) \tilde{A}_d^j(\vec{q})$

Use $\sigma^i \sigma^j = \delta^{ij} + i \epsilon^{ijk} \sigma^k$ (summed over repeated indices)

$$iM \approx ie \frac{2m}{2m} \xi'^{\dagger} (\vec{p}' + \vec{p}) \cdot \vec{A}_c(\vec{q}) \xi F_1(0)$$

$$+ ie \frac{2m}{2m} \xi'^{\dagger} (i \epsilon^{ijk} q^j \sigma^k) \xi \vec{A}_c^i(\vec{q}) F_1(0)$$

The first term, w/ $\vec{p}' \approx \vec{p}$, corresponds to the $(\vec{p} \cdot \vec{A} + \vec{A} \cdot \vec{p})$ term in the nonrelativistic Hamiltonian.

The second term contributes to the electron's magnetic moment interaction:

The magnetic field is $\vec{B} = \nabla \times \vec{A}_c$. Fourier transform:

$$\begin{aligned} \tilde{B}^k(\vec{q}) &= \int d^3x e^{+i\vec{q} \cdot \vec{x}} \epsilon^{kij} \partial_i A_{c,j}(\vec{x}) \\ &= +i q_i \epsilon^{kij} \int d^3x e^{-i\vec{q} \cdot \vec{x}} A_{c,j}(\vec{x}) \quad (\text{integrate by parts}) \\ &= -i \epsilon^{kij} q^i \tilde{A}_c^j(\vec{q}) \end{aligned}$$

Then the second term in iM above is:

$$iM \approx -ie \xi'^{\dagger} \sigma^k \xi \tilde{B}^k(\vec{q}) F_1(0)$$

This is the Born approximation to scattering off a potential,

$$V(\vec{x}) = -\vec{\mu}_i \cdot \vec{B} \quad , \quad \boxed{\vec{\mu}_i = \frac{e}{m} F_1(0) \xi'^{\dagger} \frac{\vec{\sigma}}{2} \xi}$$

The $F_2(q^2)$ term in iM contributes similarly:

$$iM \supset + \frac{e}{2m} \bar{u}(p') \frac{i}{2} [\sigma^i, \gamma^k] q^k F_2(q^2) u(p) \tilde{A}_{cl}^j(\vec{q})$$

Use again $\sigma^i \sigma^j = \delta^{ij} + i \epsilon^{ijk} \sigma^k$
 $\rightarrow [\sigma^i, \sigma^j] = 2i \epsilon^{ijk} \sigma^k$

Then $\bar{u}(p') \sigma^{jk} q^k u(p) \approx 2m \xi'^t \epsilon^{jki} q^k \sigma^i \xi$

$$iM \approx \frac{e}{2m} \cdot 2m \xi'^t \sigma^i \xi \epsilon^{jki} q^k F_2(q^2) \tilde{A}_{cl}^j(\vec{q})$$

$$= -ie \xi'^t \sigma^i \xi \tilde{B}^i(\vec{q}) F_2(q^2)$$

Comparing again w/ the Born approximation, we get another contribution to the magnetic moment,

$$\vec{\mu}_2 = \frac{e}{m} F_2(0) \xi'^t \frac{\vec{\sigma}}{2} \xi$$

where we set $q^2=0$ in $F_2(q^2)$ because we assume $\tilde{A}_{cl}^j(\vec{q})$ is dominated by $\vec{q} \approx \vec{0}$.

Adding everything together, we find a magnetic moment interaction,

$$V(\vec{x}) = -\vec{\mu} \cdot \vec{B}(\vec{x})$$

$$\vec{\mu} = \frac{e}{m} [F_1(0) + F_2(0)] \xi'^t \frac{\vec{\sigma}}{2} \xi$$

$\left\{ \frac{e}{2m} \vec{S} \right\}$ is the electron spin \vec{S} . It is common to write the electron magnetic moment in the form,

$$\vec{\mu} = g \left(\frac{e}{2m} \right) \vec{S}, \quad g = \text{Landé } g\text{-factor.}$$

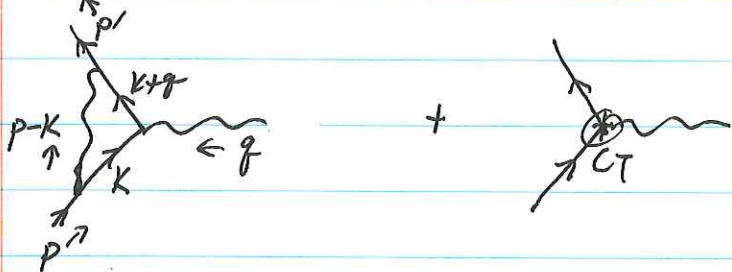
Since we have argued that $F_1(0) = 1$, we have derived,

$$\boxed{g = 2 + 2 F_2(0)}$$

$g = 2$ was the prediction of the Dirac equation. At tree level $F_2(0) = 0$, so $F_2(0) = \mathcal{O}(\alpha)$.

The value of $(g - 2)$ is called the anomalous magnetic moment of the electron.

What we need to calculate is



The counterterm is chosen so that the electron charge is e , so we expect the same counterterm which set $\frac{dZ}{d\mu} \Big|_{\mu=0} = 0$ for the electron self energy to also set $F_1(0) = 1$. This is a consequence of a Ward Identity, but we will have to wait to see this.