Phys 722

Last semester you learned how to define a <u>relativistic quantum field theory</u>, how to calculate <u>scattering amplitudes</u> in perturbation theory, and how to relate the scattering amplitudes to <u>cross sections</u> and <u>decay widths</u>.

This semester will focus on techniques for calculating <u>loop corrections</u> to scattering amplitudes, <u>renormalization</u>, <u>functional integral quantization</u>, and <u>gauge theories</u>.

A prototypical field theory for studying each of these topics is Quantum Electrodynamics. Recall the QED Feynman rules:

<u>External Lines</u>

r_A, p_A $\uparrow = u^{r_A}(p_A)$ ingoing e^-

r_1, p_1 $\uparrow = \bar{u}^{r_1}(p_1)$ outgoing e^-

r_A, p_A $\downarrow = \bar{v}^{r_A}(p_A)$ ingoing e^+

r_1, p_1 $\downarrow = v^{r_1}(p_1)$ outgoing e^+

<u>Convention</u>: For e^-, momentum is labeled along charge flow.
For e^+, momentum is labeled opposite charge flow.

 $= \varepsilon_\mu^{(r)}(p)$

μ, r, p ingoing photon

μ, r, p

 $= \varepsilon_\mu^{(r)}(p)^*$

outgoing photon

Propagators (internal lines)

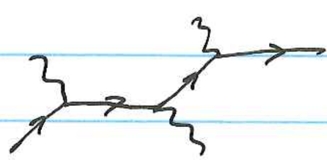
Photons:  $\dfrac{-i g_{\mu\nu}}{k^2 + i\epsilon}$ (will be justified later)

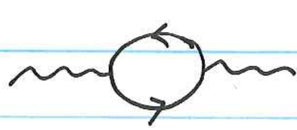
Fermions:  $\dfrac{i(\not{k} + m)}{k^2 - m^2 + i\epsilon}$

Vertex

 $-ie\gamma^\mu$

These rules are enough to calculate <u>any</u> <u>tree-level</u> diagram in QED, i.e. any diagram with no closed loops. We will deal with loops soon.

 Tree-level

 One-loop

The Coulomb Potential

In the Center-of-Mass frame two-body scattering can be interpreted as scattering off of a potential. If we take the non-relativistic limit of a scattering cross section, we can determine the potential by comparison w/ the Born approximation in non-relativistic QM:

The sol'n to the Schrödinger eqn w/ potential $V(\vec{r})$ which looks like a plane wave when $V \to 0$ is approximately

$$\psi(\vec{r}) \approx \frac{1}{(2\pi)^{3/2}}\left[e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r} f(\vec{k},\vec{k}')\right] \quad , \quad k \equiv |\vec{k}| = |\vec{k}'|$$

$$\text{where } f(\vec{k},\vec{k}') \approx -\frac{m}{2\pi}\int d^3\vec{r}'\, e^{i(\vec{k}-\vec{k}')\cdot\vec{r}'}\, V(\vec{r}')$$

$$\equiv -\frac{m}{2\pi}\tilde{V}(\vec{k}-\vec{k}')$$

and $\vec{k}' \equiv |\vec{k}|\,\hat{r}$.

In terms of $f(\vec{k},\vec{k}')$ the differential cross section is,

$$\boxed{\frac{d\sigma}{d\Omega} = |f(\vec{k},\vec{k}')|^2 = \frac{m^2}{4\pi^2}|\tilde{V}(\vec{k}-\vec{k}')|^2}$$

If $V(\vec{r})$ describes the potential between two particles that are scattering off of one another, then m should be replaced by the reduced mass $\frac{m_1 m_2}{m_1+m_2} = \frac{m}{2}$, if $m_1 = m_2 = m$.

The cross section in NRQM should be compared with our field theoretic expression,

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 \omega_{TOT}^2} \frac{|\vec{P}_1|}{|\vec{P}_A|} |M|^2$$

$$\approx \frac{1}{64\pi^2 (2m)^2} |M|^2$$

in the non-rel. limit of elastic scattering of particles with equal mass m.

Comparing the two expressions, we identify

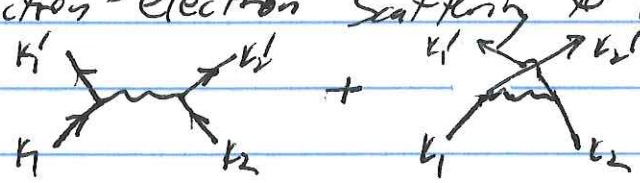
$$\boxed{|\tilde{V}(\vec{k}-\vec{k}')|^2 \approx \frac{1}{16m^4} |M(\vec{k} \to \vec{k}')|^2}$$

We could have done better by comparing non-rel. scattering amplitudes directly, which would have given

$$\boxed{\tilde{V}(\vec{k}-\vec{k}') \approx -\frac{1}{4m^2} M(\vec{k} \to \vec{k}')}$$

This expression is more useful because it carries direct information about the <u>sign</u> of the potential.

For electron-electron scattering to lowest order e^2

$iM =$ 

$$= (-ie)^2 \bar{u}(k_1')\gamma^\mu u(k_1) \frac{-ig_{\mu\nu}}{(k_1-k_1')^2} \bar{u}(k_2')\gamma^\nu u(k_2) - (k_1' \leftrightarrow k_2')$$

Consider the first term in iM.

In the nonrel limit $\bar{u}(k_1')\gamma^0 u(k_1) = u^\dagger(k_1')u(k_1)$
$\approx 2m\, \xi_1'^\dagger \xi_1$

where ξ_1 and ξ_1' are the 2-component Pauli spinors describing the initial and final state electron labeled 1.

Also, $\bar{u}(k_1')\gamma^i u(k_1) \to 0$ if $\vec{k}_1, \vec{k}_1' \to 0$.

The nonrelativistic scattering amplitude is,
(ignoring the crossed term)

$$iM = \frac{ie^2}{-|\vec{k}'-\vec{k}|^2}\left(2m\, \xi_1'^\dagger \xi_1\right)\left(2m\, \xi_2'^\dagger \xi_2\right) g_{00}$$

$$= \frac{-ie^2}{|\vec{k}'-\vec{k}|^2} \cdot 4m^2 \left(\xi_1'^\dagger \xi_1\right)\left(\xi_2'^\dagger \xi_2\right)$$

The spin of each electron is conserved, since

$$\xi_1'^\dagger \xi_1 \equiv \xi^{r_1'\dagger}\xi^{r_1} = \delta^{r_1' r_1}$$
$$\xi_2'^\dagger \xi_2 \equiv \xi^{r_2'\dagger}\xi^{r_2} = \delta^{r_2' r_2}$$

Factoring out the spin-conserving Kronecker δ's and comparing w/ the Born approx, we get the Fourier transformed potential,

$$\boxed{\tilde{V}(\vec{k}-\vec{k}') = \frac{e^2}{|\vec{k}'-\vec{k}|^2}}$$

Fourier transforming we get the potential in coordinate space. We will kill two birds w/ one stone and consider a more general potential $\tilde{V}(\vec{k}) = \frac{e^2}{|\vec{k}|^2 + m^2}$

Then, $V(\vec{r}) = e^2 \int \frac{d^3k}{(2\pi)^3} \frac{e^{-i\vec{k}\cdot\vec{r}}}{\vec{k}^2 + m^2}$

$$= e^2 \int_0^\infty dk \int_0^\pi d\theta \int_0^{2\pi} d\phi \frac{k^2}{(2\pi)^3} \sin\theta \frac{e^{-ik|\vec{r}|\cos\theta}}{k^2 + m^2}$$

$$= \frac{e^2}{(2\pi)^2} \int_0^\infty dk \frac{k^2}{k^2+m^2} \frac{e^{-ik|\vec{r}|} - e^{ik|\vec{r}|}}{-ik|\vec{r}|}$$

$$= \frac{ie^2}{4\pi^2|\vec{r}|} \int_{-\infty}^{\infty} dk \frac{k e^{-ik|\vec{r}|}}{k^2+m^2}$$

Analytically continuing to the complex k-plane and closing the integration contour in the lower half plane,

$$V(\vec{r}) = \frac{ie^2}{4\pi^2|\vec{r}|}(-2\pi i) \frac{(-im)e^{-i(-im)|\vec{r}|}}{-2im}$$

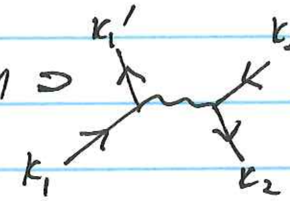
$$\boxed{V(\vec{r}) = \frac{e^2}{4\pi|\vec{r}|} e^{-m|\vec{r}|}}$$ Yukawa potential

As $m \to 0$ we recover the Coulomb potential,

$$\boxed{V(\vec{r}) = \frac{e^2}{4\pi|\vec{r}|}}$$

The sign of the potential teaches us that particles w/ the same charge <u>repel</u> in QED.

What about particle - antiparticle scattering?

$i\mathcal{M} \supset$ 

$= -(-ie)^2 \bar{u}(k_1')\gamma^\mu u(k_1) \dfrac{-ig_{\mu\nu}}{(k_1'-k_1)^2} \bar{v}(k_2)\gamma^\nu v(k_2')$

$$\bar{v}(k_2)\gamma^0 v(k_2') = v^\dagger(k_2) v(k_2') \approx 2m\, \xi_2^\dagger \xi_2' ,$$

just as for particles w/ $v \to u$.

The extra $-$ve sign came from exchanges of fermion fields:

$$\langle k_1', k_2' | : \bar{\psi} \psi \, \bar{\psi} \psi : | k_1, k_2 \rangle$$

$$\propto \langle 0 | b_{k_2'} a_{k_1'} \, \bar{\psi} \psi \, \bar{\psi} \psi \, a^\dagger_{k_1} b^\dagger_{k_2} | 0 \rangle$$

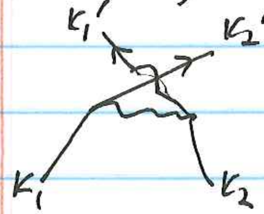
— Requires odd # exchanges of fermion ops to move the ψ's and $\bar{\psi}$'s next to the corresponding creation or annihilation operators $\to$ gives an overall (-1).

Hence, the potential between particles and antiparticles takes the same form as the potential betw particles - particles (or antiparticles - antiparticles), but w/ the opposite sign.

$\rightarrow$ $\boxed{\text{particles and antiparticles attract in QED}}$

Exchange Potentials

In the derivation of the Coulomb potential we quietly ignored one of the contributions to the scattering amplitude,



, which is equal to our previous term w/ $k_1' \leftrightarrow k_2'$, up to a sign.

Hence, we get another term in the potential

$\vec{k}_2' = -\vec{k}_1'$ in COM frame.

$$\tilde{V}_{exch.}(\vec{k}_1 - \vec{k}_2') = \tilde{V}_{exch}(\vec{k}_1 + \vec{k}_1')$$

$$= \frac{-e^2}{|\vec{k}_1 + \vec{k}_1'|^2}$$

We would not have gotten this term had we considered instead scattering of non-identical particles.

This is the exchange potential in the Hamiltonian of non-relativistic QM w/ two identical particles,

$$H = H_0 + \underset{\text{Coulomb Potential}}{V} + \underbrace{VE}_{\text{Exchange Coulomb potential}} \qquad \underset{\text{Exchange operator}}{E}|\vec{r}_1, \vec{r}_2\rangle = \underset{\text{Because these are fermions.}}{-}|\vec{r}_2, \vec{r}_1\rangle$$