Physics 722, Spring 2008
Problem Set 7: C,P,T and Group Theory
Due Thursday, April 10.

1. *C, P and T*

a) Determine the action of C, P and T on the fermion bilinears $\overline{\psi}\psi$, $i\overline{\psi}\gamma^5\psi$, $\overline{\psi}\gamma^\mu\psi$, $\overline{\psi}\gamma^\mu\gamma^5\psi$, and $i\overline{\psi}[\gamma^\mu, \gamma^\nu]\psi$.

b) Convince yourself that any Lorentz invariant formed by fermion bilinears and/or space-time derivatives is invariant under the combined transformation CPT.

c) Convince yourself that QED is invariant under C, P, and T independently. How does the electromagnetic field A_μ transform?

d) What are the discrete symmetries of the following Lagrangians? (Write the transformations of the fields which leave the action invariant.)

$$\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ - m)\psi + \frac{1}{2}(\partial_\mu\phi)^2 + ig\overline{\psi}\gamma^5\psi\phi$$

$$\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ - m)\psi + \frac{1}{2}(\partial_\mu\phi)^2 + ig\overline{\psi}\gamma^5\psi\phi - \frac{M^2}{2}\phi^2 + \frac{g_3}{3!}\phi^3$$

$$\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ - m - eA\!\!\!/)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + ig\overline{\psi}[\gamma^\mu, \gamma^\nu]\psi F_{\mu\nu}$$

2. *Lie Groups*

a) Show that if the generators of a Lie group satisfy the Lie algebra,

$$\left[T^a, T^b\right] = i\, f^{abc}\, T^c,$$

then the generators in the adjoint representation, defined by

$$(T^a)_{bc} = -i\, f^{abc},$$

satisfy the Lie algebra.

b) Show that $T^2 = T^a T^a$ commutes with all the group generators. As a consequence, $(T^2)_{ij} = C_2(r)\,\delta_{ij}$, where the constant $C_2(r)$ is called the *quadratic Casimir* of the representation r.

c) Suppose the generators are normalized so that

$$\mathrm{Tr}\, T^a T^b = C(r)\,\delta^{ab},$$

where the constant $C(r)$ depends on the representation r. Suppose the generators in the representation r are $d(r) \times d(r)$ matrices. Show that

$$d(r)C_2(r) = d(G)C(r),$$

where G stands for the adjoint representation.

d) Suppose the generators of SU(N) in the fundamental representation are normalized by

$$\mathrm{Tr}\, T^a T^b = \frac{1}{2}\delta^{ab}.$$

Calculate the quadratic Casimir C_2 in this representation.

e) Suppose the generators of SU(2) in the fundamental representation are normalized as in part (d). What is the quadratic Casimir C_2 in the d-dimensional representation of SU(2) as a function of d?

Challenge: Assume the generators of SU(N) in the fundamental representation are as in part (d). Show that in the adjoint representation $C(G) = C_2(G) = N$.