

**Physics 722, Spring 2008**  
**Final Exam**

Due Wednesday, April 30, 5pm.

Rules for the exam: You may use notes you have taken, notes I have given you, and Peskin & Schroeder's text. That is all you may use, out of fairness to your classmates. You should not discuss the problems with anyone except the instructor during the exam.

There are two problems. Each problem is worth 50 points.

Feel free to ask questions during the exam. My office phone is 221-3763. My cell phone is 272-2697. I will post answers to questions asked at the course website, <http://physics.wm.edu/~erlich/722S07>

## 1. QED $\beta$ -function with charged scalars

In class we calculated the one-loop  $\beta$ -function for the electromagnetic coupling in QED by studying the photon self energy. Here you will generalize the calculation to QED coupled to both scalar and spinor fields.

a) Write the Lagrangian for electromagnetism minimally coupled to  $N_s$  complex scalars with charges  $Q_i^{(s)}$  and masses  $m_i^{(s)}$ ,  $i = 1, \dots, N_s$ ; and  $N_f$  Dirac spinors with charges  $Q_i^{(f)}$  and masses  $m_i^{(f)}$ ,  $i = 1, \dots, N_f$ .

b) What are the Feynman rules for this theory in Feynman gauge, using the naive Feynman rules for derivative interactions?

c) Calculate the one-loop photon self energy  $\Pi_{\mu\nu}(q)$ , using dimensional regularization and imposing a physical renormalization condition. Note that there are two classes of diagrams with scalar loops that contribute (plus counterterms). You may find it useful to combine the two diagrams with scalar loops early in your calculation.

d) Gauge invariance requires that  $\Pi_{\mu\nu}(q)$  be transverse,  $q_\mu \Pi^{\mu\nu}(q) = 0$ . Is your calculation consistent with this condition?

e) Evaluate the  $\beta$ -function in this theory for small coupling from the behavior of the self energy at large momentum transfer  $q^2$ .

f) Does the sign of the  $\beta$ -function depend on  $N_f$ ,  $N_s$  and the charges  $Q_i^{(f)}$  and  $Q_i^{(s)}$ ?

## 2. Functional integral and potential energy

a) Consider a real scalar field  $\phi(x)$  coupled to a static background source  $\rho(\mathbf{x})$ , with action

$$S[\rho(\mathbf{x})] = \int d^4x \left[ \frac{1}{2} (\partial_\mu \phi)^2 - \frac{m^2}{2} \phi^2 - \rho(\mathbf{x}) \phi(x) \right].$$

It follows from our discussion of the functional integral that

$$\lim_{T \rightarrow \infty(1-i\epsilon)} \langle 0 | \exp[-iHT] | 0 \rangle = \frac{\int \mathcal{D}\phi \exp[iS[\rho(\mathbf{x})]]}{\int \mathcal{D}\phi \exp[iS[0] ]}.$$

From this relation, calculate the dependence of the ground state energy on the source  $\rho(\mathbf{x})$ , which should take the form

$$H_0 = \frac{1}{2} \int d^3x d^3x' \rho(\mathbf{x}') V(\mathbf{x}' - \mathbf{x}) \rho(\mathbf{x}).$$

You should evaluate  $V(\mathbf{x}' - \mathbf{x})$ . Don't just leave it in the form of a Fourier transform.

b) The action for electromagnetism coupled to a background current is given by

$$S = -\frac{1}{4} \int d^4x F_{\mu\nu} F^{\mu\nu} - \int d^4x J^\mu A_\mu.$$

Derive the Coulomb potential by analogy with part (a) of this problem.

*Notes for part (b):* You can assume the current is conserved,  $\partial_\mu J^\mu = 0$ . The charge density is the time component of the current. You should average over generalized Lorenz gauges as usual to simplify the functional integral.