Problem Set 7: Isospin, Yang-Mills Theory

Due Thursday, April 19.

1. *Pion-Nucleon Interactions*

The pions are pseudoscalar mesons that form a triplet under SU(2) isospin. The proton and neutron form an SU(2) doublet of Dirac ferions. In this problem you will consider a revised version of the Yukawa interactions introduced last semester that takes into account the (approximate) isospin symmetry.

Define the nucleon doublet,

$$N = \begin{pmatrix} p \\ n \end{pmatrix},$$

which transforms under an SU(2) transformation by $N \to g(\theta^a)N$, where $g(\theta^a) = \exp[i\theta^a\sigma^a/2]$ and σ^a, $a = 1, 2, 3$ are the Pauli sigma matrices.

The pions form a triplet π^a, $a = 1, 2, 3$, and we define,

$$\pi = \pi^a \frac{\sigma^a}{2},$$

which transforms as $\pi \to g\pi g^{-1}$.

Consider the theory described by the Lagrangian,

$$\mathcal{L} = \overline{N}(i\partial\!\!\!/ - m)N + \mathrm{Tr}\,(\partial_\mu\pi)(\partial^\mu\pi) - ig\,\overline{N}\gamma^5\pi N.$$

a) Show that $\mathcal{L}$ is invariant under SU(2) isospin transformations.

b) Expand the Lagrangian in components, *i.e.*

$$\mathcal{L} = \overline{p}(i\partial\!\!\!/ - m)p + \cdots - \frac{ig}{2}\,\overline{p}\gamma^5 p\pi^3 + \cdots$$

c) Define $\pi^0 = \pi^3$ and $\pi^\pm = (\pi^1 \mp i\pi^2)/\sqrt{2}$. Write $\mathcal{L}$ in terms of p, n, π^0 and $\pi^\pm$.

d) Calculate the 1PI pion self energy diagrams $\Pi^{ab}(k^2)$ contributing to the Fourier transform of $\langle 0|T(\pi^a(x)\pi^b(0))|0\rangle$. You do not need to do the momentum integrals. You should evaluate all group theoretic factors, and leave your result in terms of the 1-loop scalar self energy diagram $\Pi(k^2)$ for a single real scalar ϕ and a single Dirac fermion ψ, Yukawa coupled with Lagrangian,

$$\mathcal{L}_1 = \overline{\psi}(i\not{\partial} - m)\psi + \frac{1}{2}(\partial_\mu \phi)^2 - ig\overline{\psi}\gamma^5\psi\phi.$$

e) Do the same for the nucleon self energy $\Sigma_{ij}(\not{k})$ contributing to the Fourier transform of $\langle 0|T(N_i(x)\overline{N}_j(0))|0\rangle$, in terms of the 1-loop fermion self energy diagram $\Sigma(\not{k})$ in the theory described by $\mathcal{L}_1$ above.

2. *Yang-Mills Theory*

Consider the theory described in Problem 1, but now imagine that the SU(2) isospin symmetry is promoted to a gauge invariance. You must introduce a triplet of gauge fields A^a_μ, $a = 1, 2, 3$ to make this possible.

a) What is the gauge invariant version of the Lagrangian $\mathcal{L}$ of Problem 1? Be careful to distinguish the fundamental and adjoint representations of the generators of the gauge group, and define any symbols that you introduce.

b) What are the Feynman rules for the new vertices appearing in this theory?

c) What additional Feynman diagrams contribute to $\Pi^{ab}(k^2)$ at 1-loop?

d) What additional Feynman diagrams contribute to $\Sigma_{ij}(\not{k})$ at 1-loop?

In parts (c) and (d) you do not need to evaluate the Feynman diagrams, just draw them. You may want to revisit this problem after we have discussed ghost fields.