Physics 722, Spring 2007 Josh Erlich

Problem Set 6: Renormalizability, Non-Abelian Symmetries

Due Tuesday, April 10.

1. *Renormalizability*

We argued that in four spacetime dimensions, a theory of spin-0 and spin-1/2 fields is generally not renormalizable if the Lagrangian contains operators of mass dimension > 4, and is renormalizable if it contains all operators of mass dimension ≤ 4. What is the corresponding statement in d spacetime dimensions?

2. *Current Algebra*

We have seen a number of examples of symmetries generated by the corresponding conserved charges. For example, translations are generated by the momentum operator, and rotations are generated by the angular momentum. Those charges satisfy the algebra of the corresponding symmetry group. For example, the angular momentum operators satisfy the SO(3) algebra:

$$[L_j, L_k] = i\,\epsilon_{jkl}\,L_l.$$

The fact that conserved charges satisfy the corresponding symmetry algebra is generic. Here you will study a generic theory of scalar fields with a continuous global symmetry.

Consider the Lagrangian,

$$\mathcal{L} = \sum_{i=1}^{N} \partial_\mu \phi_i^\dagger\, \partial^\mu \phi_i - V(\phi_i),$$

where $V(\phi_i)$ is a Hermitian function of ϕ_i and $\phi_i^\dagger$ without spacetime derivatives.

Suppose $\mathcal{L}$ has a group of symmetries parametrized by $\dim G$ parameters θ^a,

$$\phi_j \to \phi_j + i\,\theta^a\,T^a_{jk}\,\phi_k + \mathcal{O}(\theta^2),$$

generated by the $N \times N$ matrices T^a, $a = 1, \ldots, \dim G$, satisfying the algebra,

$$\left[T^a, T^b\right] = i\,f^{abc}\,T^c.$$

a) What are the conserved currents due to the symmetries? What are the corresponding charges?

b) Using the equal-time commutation relations,

$$[\pi_i(\mathbf{x},t), \phi_j(\mathbf{y},t)] = -i\,\delta_{ij}\,\delta^3(\mathbf{x}-\mathbf{y}),$$

show that the currents satisfy the *current algebra*,

$$\left[J_0^a(\mathbf{x},t), J_0^b(\mathbf{y},t)\right] = i\,f^{abc}\,J_0^c(\mathbf{x},t)\,\delta^3(\mathbf{x}-\mathbf{y}).$$

c) Show that $\left[Q^a, J_0^b(\mathbf{x},t)\right] = i\,f^{abc}\,J_0^c(\mathbf{x},t)$.

d) Show that the charges satisfy the *charge algebra*,

$$\left[Q^a, Q^b\right] = i\,f^{abc}\,Q^c.$$

Comments:
In your free time you can check that the same relations hold in a theory of fermions, or in a theory with both fermions and scalars. The current algebra depends only on the symmetry group, not on the details of the underlying theory.

Current algebra provides a powerful tool for deriving selection rules and relating the scattering amplitudes of different processes related by symmetry transformations, as we have seen in the discussion of isospin in class.

3. *Higgs Yukawa couplings*

Suppose $\Phi = (\phi_1^+, \phi_2^0)$ is a pair of complex scalar fields that transform as a doublet under an SU(2) symmetry. Suppose $\Psi_L = (\psi_{L1}^q, \psi_{L2}^{q-1})$ is a pair of fermions with left-handed chirality that transform as a doublet under the same SU(2) symmetry; and ψ_{R1}^q and ψ_{R2}^{q-1} are two right-handed SU(2) singlets. The superscripts stand for the electric charge.

a) Write down *all* cubic interactions among these various fields that are invariant under the SU(2) symmetry and electromagnetism. Recall that if Φ transforms as an SU(2) doublet, then so does $i\,\sigma^2\,\Phi^*$, but the electric charges change sign in the process (because of the complex conjugation). Remember that the interactions should be Hermitian and Lorentz invariant.

b) Replace Φ by $(0, v)$ in the interactions you constructed in part (a), for some constant v. Expand those terms in terms of ψ_{Li} and ψ_{Ri}. You should find terms that look like mass terms for all of the fields.

c) Assume ψ_{R1} does not exist. Which fields are massless?

Comments:
The Higgs field transforms as a doublet under the $\mathrm{SU}(2)_W$ gauge symmetry of the Standard Model. It is assumed that the Higgs field obtains a vacuum expectation value that can be chosen (by a symmetry transformation) to take the form in part (b). As a result, the Yukawa couplings you constructed in part (a) give rise to fermion masses. It is important that SU(2) has the pseudo-reality property that $i\,\sigma^2\,\Phi^*$ transforms as a doublet if Φ does.

If there is no right-handed neutrino, then the neutrino cannot get simple Yukawa couplings to the Higgs and the mass must come from elsewhere. Neutrinos can have Majorana masses without there being right-handed neutrinos, so despite the experimental evidence for neutrino masses it is not known whether or not there are right-handed neutrinos.