Physics 722, Spring 2007 Josh Erlich

Problem Set 4: Gamma matrices in arbitrary dimensions, due Tuesday, March 6.

In dimensional regularization we analytically continue in the number of spacetime dimensions. It is natural to consider what the gamma matrices look like in other than four (integer) dimensions.

1. *Even Dimensions*

In d spacetime dimensions, we want to find d matrices satisfying:

$$\gamma_0^\dagger = \gamma_0, \quad \gamma_i^\dagger = -\gamma_i, \quad i = 1, \ldots, d-1,$$

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}, \quad \mu, \nu = 0, \ldots, d-1.$$

It's easier to construct matrices satisfying,

$$\gamma_\mu^\dagger = \gamma_\mu, \quad \mu = 1, \ldots, d,$$

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \quad \mu, \nu = 1, \ldots, d.$$

You can get the gamma matrices you want from these by letting $\gamma_d \to \gamma_0,\ \gamma_j \to i\gamma_j$.

a) Assume d is even. Define,

$$\begin{aligned} a_1 &= \frac{1}{2}(\gamma_1 + i\gamma_2), \\ a_2 &= \frac{1}{2}(\gamma_3 + i\gamma_4), \\ &\vdots \\ a_{d/2} &= \frac{1}{2}(\gamma_{d-1} + i\gamma_d), \end{aligned}$$

where the gamma matrices in these expressions satisfy the second set of conditions above.

Show that,

$$\{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0,$$

$$\{a_i, a_j^\dagger\} = \delta_{ij} \quad i, j = 1, \ldots, d/2.$$

This is the algebra of raising and lowering operators for $d/2$ independent two-level systems.

b) In two dimensions, construct a matrix representation for a and $a^\dagger$. What are γ_1 and γ_2 in that representation?

c) In d even dimensions, what is the dimensionality of your representation of the gamma matrices? Evaluate $\mathrm{Tr}\,1$ and $\mathrm{Tr}\,\not{p}\not{p}$ in that representation.

d) Check that $\prod_i \gamma_i$ anticommutes with all of the γ_i. This is the analog of γ_5 in any even dimension.

2. *Odd dimensions*

In odd dimensions the first $d-1$ gamma matrices can be constructed as above.

a) Show that $\gamma_d = \pm\gamma_1\gamma_2\cdots\gamma_{d-1}$ completes the gamma matrix algebra.

The condition $\gamma_d^\dagger = \gamma_d$ only determines γ_d up to a sign. Hence, there are two different representations of the gamma matrices in odd dimensions, exchanged by parity.

b) In a parity conserving theory you need both spin-up and spin-down particles, so you need to include both representations of the gamma matrices. For odd d, what is the dimensionality of the gamma matrices you have constructed in a parity conserving theory?

What is $\mathrm{Tr}\,1$ and $\mathrm{Tr}\,\not{p}\not{p}$ in this representation?