Problem Set 6: Hyperscaling Relations
Due Thursday, April 6.

1. *Relations Between Critical Exponents*

As a result of the homogeneity of the free energy, in the Landau-Ginzburg theory of magnetization the singular part of the free energy near the critical point is of the form,

$$f(t,h) \approx t^{2-\alpha}\, g_f(h/t^{\Delta}),$$

for some critical exponent α, gap exponent Δ anf function $g_f(x)$.

Consider the saddle point approximation in which,

$$f \simeq -\frac{t^2}{u} \quad \text{for } h = 0,\ t < 0,$$

$$f \simeq -\frac{h^{4/3}}{u^{1/3}} \quad \text{for } h \neq 0,\ t = 0.$$

a) Show that $\alpha = 0$ and $\Delta = 3/2$ in the saddle point approximation. The exponent α is consistent with the definition of the critical exponent associated with the heat capacity,

$$C \sim -\frac{\partial^2 f}{\partial t^2}.$$

b) The critical exponent β is defined by $m(t, h = 0) \sim t^{\beta}$, where the magnetization is given by,

$$m(t,h) \sim \frac{\partial f}{\partial h}.$$

Express β in terms of α and Δ.

c) As $t \to 0$ the magnetization behaves as,

$$m(t \to 0, h) \sim t^{\beta} \left(\frac{h}{t^{\Delta}}\right)^{\delta},$$

which defines the critical exponent δ.

Express δ in terms of Δ and β.

the magnetic susceptibility:

$$\chi(t,h) \sim \frac{\partial m}{\partial h} \sim t^{-\gamma} \text{ if h} = 0.$$

Express γ in terms of Δ and α.

e) The correlation length also has homogeneous behavior near the critical point,

$$\xi(t,h) \sim t^{-\nu} g(h/t^{\Delta}),$$

for some exponent ν and function $g(x)$, with the same gap exponent Δ. The free energy scales with ξ as,

$$f(t,h) \sim \frac{\log\ Z}{L^d} \sim \xi^{-d}.$$

Derive the following hyperscaling relations:

Rushbrooke's identity: $\alpha + 2\beta + \gamma = 2$;

Widom's identity: $\delta - 1 = \gamma/\beta$;

Josephson's identity, $2 - \alpha = d\nu$.

f) In two dimensions the critical exponents can be calculated exactly, and it is found that with one degree of freedom: $\alpha = 0,\ \beta = 1/8,\ \gamma = 7/4,\ \delta = 15,\ \nu = 1,\ \eta = 1/4$. How do these exponents compare with the predictions of the hyperscaling relations?