Phys 721 F'23 Problem Set 8 Solutions

1. Suppose $A_i(x)$ are fermionic fields.

$T(A_1) = A_1 = :A_1:$

$T(A_1 A_2) = :A_1 A_2: + \overbrace{A_1 A_2}$ by definition of the contraction.

Assume $T(A_1 \cdots A_n) = W(A_1 \cdots A_n)$

$\equiv \sum$ all normal-ordered contractions of $A_1 \cdots A_n$.

Decompose free fields in part w/ creation ops $A_i^{(-)}$, and part w/ annihilation ops $A_i^{(+)}$.

Assume w/o loss of generality $t_0 > t_1 > \cdots > t_n$, and consider $T(A_0 \cdots A_n)$.

$$T(A_0 A_1 \cdots A_n) = A_0^{(-)} W(A_1 \cdots A_n) + A_0^{(+)} W(A_1 \cdots A_n)$$

If n is odd, write

$$T(A_0 \cdots A_n) = A_0^{(-)} W(A_1 \cdots A_n) - W(A_1 \cdots A_n) A_0^{(+)} + \{A_0^{(+)}, W(A_1 \cdots A_n)\}.$$

The first two terms are the sum of all normal-ordered contractions not including contractions with A_0.

The last term can be expanded:

$$\{A_0^{(+)}, W(A_1 \cdots A_n)\} = W(\{A_0^{(+)}, A_1\} A_2 \cdots A_n) - W(A_1 \{A_0^{(+)}, A_2\} A_3 \cdots A_n) + \cdots + W(A_1 \cdots A_{n-1} \{A_0^{(+)}, A_n\})$$

This is the sum of all normal-ordered contractions that include contractions with A_0.

Hence, $T(A_0 A_1 \cdots A_n) = W(A_0 A_1 \cdots A_n)$ for odd n.

For even n, write

$$T(A_0 \cdots A_n) = A_0^{(-)} W(A_1 \cdots A_n) + W(A_1 \cdots A_n) A_0^{(+)} + [A_0^{(+)}, W(A_1 \cdots A_n)]$$

The first two terms are the sum of all normal-ordered contractions of $A_0 \cdots A_n$ not including contractions w/ A_0.

The last term can be expanded:

$$[A_0^{(+)}, W(A_1 \cdots A_n)] = W(\{A_0^{(+)}, A_1\} A_2 \cdots A_n) - W(A_1 \{A_0^{(+)}, A_2\} A_3 \cdots A_n) + \cdots - W(A_1 \cdots A_{n-1} \{A_0^{(+)}, A_n\})$$

This is the sum of all normal-ordered contractions of $A_0 \cdots A_n$ including contractions w/ A_0.

Hence, for all n, $T(A_0 \cdots A_n) = W(A_0 \cdots A_n)$. ∎

2. a) $$\langle 0|T[\phi(x_1)\phi(x_2)\phi(x_3)]|0\rangle = \phi(x_1)\phi(x_2)\cancelto{0}{\langle 0|\phi(x_3)|0\rangle} + \phi(x_1)\phi(x_3)\cancelto{0}{\langle 0|\phi(x_2)|0\rangle} + \phi(x_2)\phi(x_3)\cancelto{0}{\langle 0|\phi(x_1)|0\rangle}$$

$$= \boxed{0}.$$

$$\langle 0|T[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)]|0\rangle = \phi(x_1)\phi(x_2)\,\phi(x_3)\phi(x_4) + \phi(x_1)\phi(x_3)\,\phi(x_2)\phi(x_4) + \phi(x_1)\phi(x_4)\,\phi(x_2)\phi(x_3)$$

$$= \boxed{\Delta_F(x_1-x_2)\Delta_F(x_3-x_4) + \Delta_F(x_1-x_3)\Delta_F(x_2-x_4) + \Delta_F(x_1-x_4)\Delta_F(x_2-x_3)}.$$

b) For complex ϕ, $\langle 0|T[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)]|0\rangle = \boxed{0}$

because $\phi(x_1)\phi(x_2) = 0$.

$$\langle 0|T[\phi(x_1)\phi(x_2)\phi(x_3)^\dagger\phi(x_4)^\dagger]|0\rangle = \phi(x_2)\phi(x_3)^\dagger\,\phi(x_1)\phi(x_4)^\dagger + \phi(x_1)\phi(x_3)^\dagger\,\phi(x_2)\phi(x_4)^\dagger$$

$$= \boxed{\Delta_F(x_2-x_3)\Delta_F(x_1-x_4) + \Delta_F(x_1-x_3)\Delta_F(x_2-x_4)}$$