Physics 721, Fall 2023
Problem Set 7 Due Monday, October 30.

Massive electrodynamics

Up to the addition of total derivatives, the most general Lagrangian density for the vector field including terms quadratic in the field with at most two derivatives is:

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu A_\nu\, \partial^\mu A^\nu + b\, \partial_\mu A_\nu \partial^\nu A^\mu\right) + c\, A_\mu A^\mu.$$

(a) What are the Euler-Lagrange equations for this theory?

(b) Assume a plane wave solution of the form $A^\mu(x) = \varepsilon^\mu(\mathbf{k})e^{-ik\cdot x}$. What are the Euler-Lagrange equations in terms of ε^μ and k^μ?

(A^μ is real, but as usual we describe two solutions at once – the real and imaginary parts of the plane wave. We can do this because the Euler-Lagrange equations are linear in this theory.)

(c) 4D Longitudinal mode: Assume $\varepsilon^\mu(\mathbf{k}) \propto k^\mu$. What are the Euler-Lagrange equations for this *ansatz* in terms of ε^μ and k^μ? Define $k_\mu k^\mu \equiv m_L^2$. What is the longitudinal mass m_L in terms of the parameters b and c in the Lagrangian?

(d) 4D Transverse modes: Repeat part (c) assuming that $\varepsilon_\mu k^\mu = 0$ (instead of $\varepsilon^\mu \propto k^\mu$). This time define $k_\mu k^\mu \equiv m_T^2$. What is m_T?

(e) The longitudinal mode will not propagate if $m_L \to \infty$. What choice of b accomplishes this? Make that choice and rewrite $\mathcal{L}$ in terms of A^μ, m_T, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

This is the Proca Lagrangian of massive electrodynamics. You should recover Maxwell's theory if you take $m_T \to 0$.

(f) Consider the Proca Lagrangian you have just derived. If $m_T \neq 0$ then the action is not gauge invariant. Show that the Lorenz gauge condition $\partial_\mu A^\mu = 0$ follows from the equations of motion as long as $m_T \neq 0$.