Physics 721, Fall 2023
Problem Set 3
Due Monday, October 2.

1. *4-momentum as generator of spacetime translations*

Consider a free real scalar field $\phi(x)$.

a) Express the normal-ordered Hamiltonian H and spatial momentum $\mathbf{P}$ as integrals involving raising and lowering operators, and using the harmonic oscillator commutation relations show that:

$$e^{iHt}\, a_{\mathbf{k}}\, e^{-iHt} = a_{\mathbf{k}} e^{-i\omega_{\mathbf{k}} t}$$
$$e^{iHt}\, a^{\dagger}_{\mathbf{k}}\, e^{-iHt} = a^{\dagger}_{\mathbf{k}}\, e^{i\omega_{\mathbf{k}} t}$$
$$e^{-i\mathbf{P}\cdot\mathbf{x}}\, a_{\mathbf{k}}\, e^{i\mathbf{P}\cdot\mathbf{x}} = a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$$
$$e^{-i\mathbf{P}\cdot\mathbf{x}}\, a^{\dagger}_{\mathbf{k}}\, e^{i\mathbf{P}\cdot\mathbf{x}} = a^{\dagger}_{\mathbf{k}}\, e^{-i\mathbf{k}\cdot\mathbf{x}}.$$

b) Using this, show that,

$$\phi(x) = e^{i(Ht-\mathbf{P}\cdot\mathbf{x})}\, \phi(0)\, e^{-i(Ht-\mathbf{P}\cdot\mathbf{x})}.$$

Hence, the 4-momentum operator $(H, \mathbf{P})$ generates spacetime translations of the field $\phi(x)$. This is a specific example of a general phenomenon: the conserved charge due to a symmetry generates the corresponding symmetry transformation on the fields.

c) Show that $\langle 0|\phi(x)\,\phi(y)|0\rangle = \langle 0|\phi(x-y)\,\phi(0)|0\rangle$.

2. *Global SO(2) symmetry of a pair of real scalars*

Suppose $\phi_1(x)$ and $\phi_2(x)$ are a pair of real scalar fields with Lagrangian,

$$\mathcal{L} = \frac{1}{2}\sum_{j=1}^{2}\left((\partial_\mu\phi_j)^2 - m^2\,\phi_j^2\right).$$

Each of the two fields has its own set of creation and annihilation operators. Decompose the fields in plane waves as,

$$\phi_j(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_{\mathbf{k}}}}\left(a^{(j)}_{\mathbf{k}}\, e^{-ik\cdot x} + a^{(j)\,\dagger}_{\mathbf{k}}\, e^{ik\cdot x}\right),$$

where $k^0 = \omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$.

Consider the symmetry transformation,

$$\begin{aligned} \phi_1 &\rightarrow \cos\theta\,\phi_1 + \sin\theta\,\phi_2 \\ \phi_2 &\rightarrow \cos\theta\,\phi_2 - \sin\theta\,\phi_1, \end{aligned}$$

for $0 \leq \theta < 2\pi$.

Calculate the normal-ordered charge associated with this symmetry in terms of a single d^3k integral involving the creation and annihilation operators.

3. *Particle number operator*

a) Consider a free real scalar field $\phi(x)$. Show that the operator

$$N \equiv \int \frac{d^3k}{(2\pi)^3}\, a_{\mathbf{k}}^\dagger a_{\mathbf{k}}$$

counts the number of particles in a state.

b) Write N in terms of integrals involving the field $\phi(x)$ and its derivatives.