Phys 721 F'23 Problem Set 1 Solutions

1. a) In the Dirac basis $i\hbar \frac{\partial \psi}{\partial t} = mc^2 \beta \psi = mc^2 \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \psi$

Eigenvectors of β are $\underbrace{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}}_{\text{Eigenvalue } +1}, \underbrace{\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}}_{\text{Eigenvalue } -1}$

Write $\psi = \begin{pmatrix}\tilde{\varphi}\\ \tilde{\chi}\end{pmatrix}$. Then $i\hbar \frac{\partial \tilde{\varphi}}{\partial t} = mc^2 \tilde{\varphi}$

$$i\hbar \frac{\partial \tilde{\chi}}{\partial t} = -mc^2 \tilde{\chi}$$

The solutions are $\tilde{\varphi} = e^{-imc^2 t/\hbar}\begin{pmatrix}1\\0\end{pmatrix}$, $\tilde{\varphi} = e^{-imc^2 t/\hbar}\begin{pmatrix}0\\1\end{pmatrix}$

$$\tilde{\chi} = e^{imc^2 t/\hbar}\begin{pmatrix}1\\0\end{pmatrix}, \quad \tilde{\chi} = e^{imc^2 t/\hbar}\begin{pmatrix}0\\1\end{pmatrix}$$

I.e. a complete set of solutions w/ definite energy are

$$\boxed{\begin{aligned} \psi_1(t) &= e^{-imc^2 t/\hbar}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, & \psi_2(t) &= e^{-imc^2 t/\hbar}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \\ \psi_3(t) &= e^{imc^2 t/\hbar}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, & \psi_4(t) &= e^{imc^2 t/\hbar}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \end{aligned}}$$

b) $\psi = e^{imc^2 t/\hbar}\begin{pmatrix}\varphi\\ \chi\end{pmatrix}$. Plugging this into the Dirac eqn,

$$\boxed{i\hbar \frac{\partial}{\partial t}\begin{pmatrix}\varphi\\ \chi\end{pmatrix} = c\vec{\sigma}\cdot\left(\vec{p} - \frac{e}{c}\vec{A}\right)\begin{pmatrix}\chi\\ \varphi\end{pmatrix} + e\phi\begin{pmatrix}\varphi\\ \chi\end{pmatrix} + 2mc^2\begin{pmatrix}\varphi\\ 0\end{pmatrix}}$$

c) From the top components in part (b),

$$\boxed{\varphi \approx -\frac{1}{2mc^2}\, c\vec{\sigma}\cdot\left(\vec{p} - \frac{e}{c}\vec{A}\right)\chi}$$

φ contains the small components of ψ.

χ contains the large components of ψ.

d) From the bottom components in part (b),

$$i\hbar \frac{\partial \chi}{\partial t} = -\frac{1}{2m}\, \vec{\sigma}\cdot\left(\vec{p} - \frac{e}{c}\vec{A}\right)\, \vec{\sigma}\cdot\left(\vec{p} - \frac{e}{c}\vec{A}\right)\chi + e\phi\chi$$

Using $(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + i\,\vec{\sigma}\cdot(\vec{a}\times\vec{b})$,

$$i\hbar \frac{\partial \chi}{\partial t} = -\frac{1}{2m}\left[\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + i\,\vec{\sigma}\cdot\left(\vec{p} - \frac{e}{c}\vec{A}\right)\times\left(\vec{p} - \frac{e}{c}\vec{A}\right)\right]\chi + e\phi\chi$$

Using $\left(\vec{p} - \frac{e}{c}\vec{A}\right)\times\left(\vec{p} - \frac{e}{c}\vec{A}\right) = -\frac{e}{c}\frac{\hbar}{i}\nabla\times\vec{A} = i\frac{e\hbar}{c}\vec{B}$,

$$i\hbar \frac{\partial \chi}{\partial t} = \left[-\frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + \frac{e\hbar}{2mc}\,\vec{\sigma}\cdot\vec{B} + e\phi\right]\chi$$

In a uniform magnetic field $\vec{A} = \frac{1}{2}\vec{B}\times\vec{r}$.

Using $\vec{r}\times\vec{p} = \vec{L}$, $\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 = \vec{p}^{\,2} - \frac{e}{c}\vec{L}\cdot\vec{B}$

$$\boxed{i\hbar \frac{\partial \chi}{\partial t} = \left[-\frac{\vec{p}^{\,2}}{2m} + \frac{e}{2mc}\left(\vec{L} + 2\vec{S}\right)\cdot\vec{B}\right]\chi}$$

Comparing with the Dirac eqn. for positive-energy solutions, and with the nonrelativistic Schrödinger equation, the Hamiltonian on the right-hand-side of the equation is scaled by -1. This would seem to suggest an instability, as the energy is unbounded below. Dirac's solution was to define the vacuum as the state in which all the negative energy modes are filled. Quantum field theory will provide an alternative interpretation.

2. a) $T^{\mu\nu} B_{\nu\mu} \to (\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T^{\alpha\beta})((\Lambda^{-1})^\gamma{}_\nu (\Lambda^{-1})^\delta{}_\mu B_{\gamma\delta})$

$$= \underbrace{(\Lambda^{-1})^\gamma{}_\nu \Lambda^\nu{}_\beta}_{\delta^\gamma{}_\beta} \underbrace{(\Lambda^{-1})^\delta{}_\mu \Lambda^\mu{}_\alpha}_{\delta^\delta{}_\alpha} T^{\alpha\beta} B_{\gamma\delta}$$

$= T^{\alpha\beta} B_{\beta\alpha} = T^{\mu\nu} B_{\nu\mu}$. This is how a scalar transforms.

$$T^{\mu\nu} B_{\mu\nu} \to (\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T^{\alpha\beta})((\Lambda^{-1})^\gamma{}_\mu (\Lambda^{-1})^\delta{}_\nu B_{\gamma\delta})$$

$$= \underbrace{(\Lambda^{-1})^\gamma{}_\mu \Lambda^\mu{}_\alpha}_{\delta^\gamma{}_\alpha} \underbrace{(\Lambda^{-1})^\delta{}_\nu \Lambda^\nu{}_\beta}_{\delta^\delta{}_\beta} T^{\alpha\beta} B_{\gamma\delta}$$

$$= T^{\alpha\beta} T_{\alpha\beta} = T^{\mu\nu} B_{\mu\nu}$$

b) $T^{\mu\nu} B_\mu{}^\alpha \to (\Lambda^\mu{}_\beta \Lambda^\nu{}_\gamma T^{\beta\gamma})((\Lambda^{-1})^\delta{}_\mu \Lambda^\alpha{}_\lambda B_\delta{}^\lambda)$

$$= \Lambda^\nu{}_\gamma \Lambda^\alpha{}_\lambda \underbrace{(\Lambda^{-1})^\delta{}_\mu \Lambda^\mu{}_\beta}_{\delta^\delta{}_\beta} T^{\beta\gamma} B_\delta{}^\lambda$$

$$= \Lambda^\nu{}_\gamma \Lambda^\alpha{}_\lambda T^{\beta\gamma} B_\beta{}^\lambda$$

c) $\partial_\mu \phi(x) \partial^\mu \phi(x) = \eta^{\mu\nu} \partial_\mu \phi(x) \partial_\nu \phi(x)$

$$\to \eta^{\mu\nu} \partial_\mu \phi(\Lambda^{-1}x) \partial_\nu \phi(\Lambda^{-1}x)$$

$$= \eta^{\mu\nu} (\Lambda^{-1})^\alpha{}_\mu \partial_\alpha \phi(\tilde{x}) (\Lambda^{-1})^\beta{}_\nu \partial_\beta \phi(\tilde{x}) \Big|_{\tilde{x}=\Lambda^{-1}x}$$

$= \eta^{\alpha\beta} \partial_\alpha \phi(\tilde{x}) \partial_\beta \phi(\tilde{x}) \Big|_{\tilde{x}=\Lambda^{-1}x}$ using $\eta^{\mu\nu}(\Lambda^{-1})^\alpha{}_\mu (\Lambda^{-1})^\beta{}_\nu = \eta^{\alpha\beta}$

$= \partial_\alpha \phi(\tilde{x}) \partial^\alpha \phi(\tilde{x})$. Hence, $\partial_\mu \phi \partial^\mu \phi$ is a Lorentz scalar.

d) Suppose $\eta_{\mu\nu}$ transforms as a tensor under Lorentz transformations, i.e.

$$\eta_{\mu\nu} \xrightarrow{\Lambda} (\Lambda^{-1})^{\alpha}{}_{\mu} (\Lambda^{-1})^{\beta}{}_{\nu} \eta_{\alpha\beta} = \eta_{\mu\nu}$$

where the equality follows from the defining property of the Lorentz transformations, recognizing that if Λ is a Lorentz transformation, then so is Λ^{-1}.

e) $x^{\mu} \xrightarrow{\Lambda} \Lambda^{\mu}{}_{\nu} x^{\nu}$

$$x_{\mu} = \eta_{\mu\nu} x^{\nu} \longrightarrow \eta_{\mu\nu} \Lambda^{\nu}{}_{\alpha} x^{\alpha}$$

$$= \eta_{\mu\nu} \Lambda^{\nu}{}_{\alpha} \underbrace{\eta^{\alpha\beta} x_{\beta}}_{x^{\alpha}} \equiv L^{\beta}{}_{\mu} x_{\beta}$$

Consider $L^{\beta}{}_{\mu} \equiv \eta_{\mu\nu} \Lambda^{\nu}{}_{\alpha} \eta^{\alpha\beta}$

$$\Lambda^{\mu}{}_{\gamma} L^{\beta}{}_{\mu} = \underbrace{\Lambda^{\mu}{}_{\gamma} \eta_{\mu\nu} \Lambda^{\nu}{}_{\alpha}}_{\eta_{\gamma\alpha} \text{ by the definition of Lorentz transfs.}} \eta^{\alpha\beta}$$

$$= \delta^{\beta}{}_{\gamma}$$

Hence, $L^{\beta}{}_{\mu} = (\Lambda^{-1})^{\beta}{}_{\mu}$, i.e. $\eta_{\mu\nu} \Lambda^{\nu}{}_{\alpha} \eta^{\alpha\beta} = (\Lambda^{-1})^{\beta}{}_{\mu}$

Finally, $x_{\mu} \xrightarrow{\Lambda} (\Lambda^{-1})^{\beta}{}_{\mu} x_{\beta}$