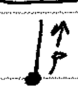
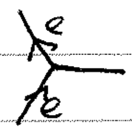
Phys 721 F'22 Problem Set 10 Solutions

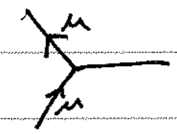
1. a) $\mathcal{L} \supset -e\,\bar{\Psi}_e \Psi_e \phi - e\,\bar{\Psi}_\mu \Psi_\mu \phi$

Feynman Rules

e or μ		e or μ		ϕ	
	$\bar{u}^r(p)$		$v^r(p)$		1
	$u^r(p)$		$\bar{v}^r(p)$		1



$-ie$



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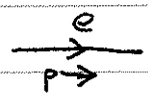
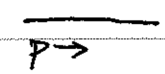
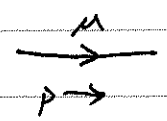


$$\frac{i(\not{p}+m_e)}{p^2-m_e^2+i\epsilon}$$

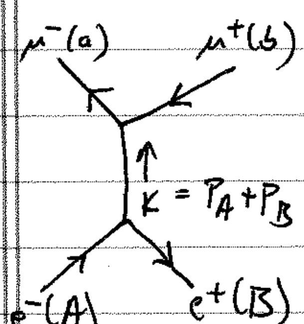


$$\frac{i}{p^2+i\epsilon}$$



$$\frac{i(\not{p}+m_\mu)}{p^2-m_\mu^2+i\epsilon}$$

b) $e^+e^- \to \mu^+\mu^-$



$$iM = \bar{u}^{r_a}(p_a)(-ie)v^{r_b}(p_b)\;\bar{v}^{r_B}(p_B)(-ie)u^{r_A}(p_A) \times \frac{i}{(p_A+p_B)^2+i\epsilon}$$

$$|M|^2 = \frac{e^4}{(p_A+p_B)^4}\left(\bar{u}^a v^b \bar{v}^b u^a\right)\left(\bar{v}^B u^A \bar{u}^A v^B\right)$$

Average over initial spins, sum over final spins:

$$\frac{1}{4}\sum_{spins}|M|^2 = \frac{e^4}{4(P_A+P_B)^4}\sum_{spins} Tr\left(u^a \bar{u}^a\, v^b \bar{v}^b\right) Tr\left(v^B \bar{v}^B u^A \bar{u}^A\right)$$

$$= \frac{e^4}{4(P_A+P_B)^4} Tr\left[(\not{p}_a + m_\mu)(\not{p}_b - m_\mu)\right] Tr\left[(\not{p}_B - \cancel{m_e})(\not{p}_A + \cancel{m_e})\right]$$

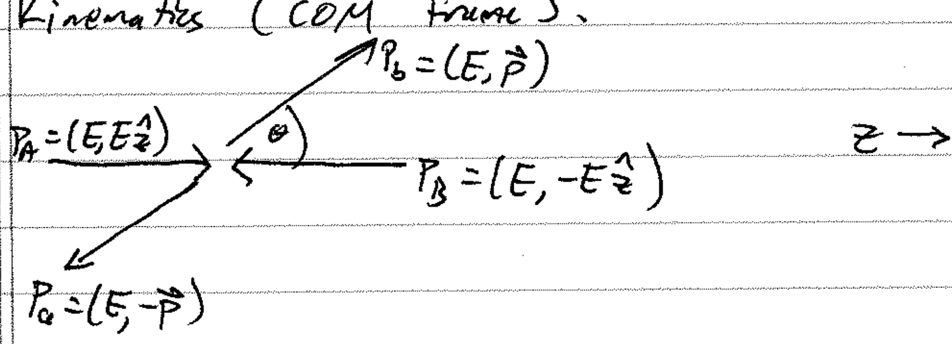
Use $Tr(\not{p}_a \not{p}_b) = p_a^\mu p_b^\nu\, Tr(\gamma_\mu \gamma_\nu) = 4\, p_a \cdot p_b$

$Tr\, \not{p} = p^\mu\, Tr\, \gamma_\mu = 0$

$Tr\, \mathbb{1} = 4$

$$\Rightarrow \frac{1}{4}\sum_{spins}|M|^2 = \frac{e^4}{4(P_A+P_B)^4}\left[4 p_a \cdot p_b - 4 m_\mu^2\right]\left[4 P_A \cdot P_B\right]$$

Kinematics (COM frame):



$(P_A+P_B)^2 = 4E^2 = \omega_{Tot}^2$, $\quad E^2 = \vec{P}_a^{\,2} + m_\mu^2$

$P_A \cdot P_B = 2E^2$

$P_a \cdot P_b = E^2 + \vec{p}^{\,2} = 2E^2 - m_\mu^2$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 \omega_{Tot}^2} \frac{|\vec{P_a}|}{|\vec{P_A}|} \sum_{spins} |M|^2 \cdot \frac{1}{4}$$

$$= \frac{e^4}{4 \cdot 64\pi^2 \cdot 4E^2} \frac{\sqrt{E^2 - m_\mu^2}}{E} \cdot \frac{4 \cdot 4}{(4E^2)^2} \left[2E^2 - m_\mu^2 - m_\mu^2\right]\left[2E^2\right]$$

$$= \boxed{\frac{\alpha^2}{16E^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left(1 - \frac{m_\mu^2}{E^2}\right)}$$

c) $\sigma_{Tot} = \int d\Omega \frac{d\sigma}{d\Omega} = 4\pi \frac{d\sigma}{d\Omega}$

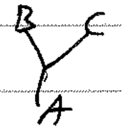
(because $\frac{d\sigma}{d\Omega}$ is independent of angles)

$$\boxed{\sigma_{Tot} = \frac{\alpha^2 \pi}{4E^2} \sqrt{1 - m_\mu^2/E^2} \left(1 - m_\mu^2/E^2\right)}$$

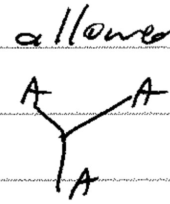
d) The crucial difference between $\frac{d\sigma}{d\Omega}$ in this theory and in QED is that there is no $\measuredangle$-dependence in this case, but there is in QED.
By measuring the $\measuredangle$-dependence of the differential cross section we learn about the spin(s) of the exchanged particle(s)

2. a) $A \to BB$ from 

vertex

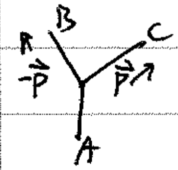
$A \to BC$ from 

vertex

$A \to AA$ is not allowed kinematically even though there is a 

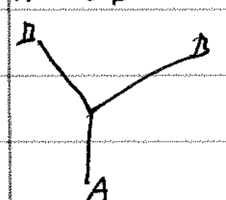
vertex

b) $A \to BC$



$i\mathcal{M} = -ig_3$, $\Gamma(A \to BC) = \dfrac{g_3^2 |\vec{p}|}{8\pi M_A^2} = \dfrac{g_3^2 \sqrt{(M_A/2)^2 - m^2}}{8\pi M_A^2}$

$A \to BB$



Each B field can be contracted with either of the two final state B's. Hence, the Feynman diagram gives

$i\mathcal{M} = -2ig_2$

$$\Gamma(A \to BB) = \frac{\overbrace{(2g_2)^2}^{\text{From } |\mathcal{M}|^2} |\vec{p}|}{\underbrace{16\pi}_{} M_A^2} = \frac{g_2^2 \sqrt{(M_A/2)^2 - m^2}}{4\pi M_A}$$

Extra factor of 2 because we only integrate over half of the 4π solid $\measuredangle$ to avoid double counting final states

$$\frac{\Gamma(A \to BC)}{\Gamma(A \to BB)} = \frac{1}{2} \frac{g_3^2}{g_2^2}$$

c) $\Gamma(A \to BC) + \Gamma(A \to BB)$

$$= \frac{(2g_2^2 + g_3^2)\sqrt{M_A^2 - 4m^2}}{16\pi M_A^2}$$

d) $\Gamma(B \to \text{anything}) = 0$ because no decays are kinematically allowed.