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More Classical Field Theory

Example: Two real scalar fields $\phi_1(x)$, $\phi_2(x)$.

Most general Lagrangian density satisfying:

1) $\mathcal{L}$ is a Lorentz scalar
2) $\mathcal{L}$ is quadratic in $\phi_1, \phi_2, \partial_\mu \phi_1, \partial_\mu \phi_2$
3) Symmetry under $\phi_1 \to \phi_1 \cos\theta + \phi_2 \sin\theta$

$\phi_2 \to -\phi_1 \sin\theta + \phi_2 \cos\theta$

(like rotation in two dimensions)

$$\boxed{\mathcal{L} = \frac{1}{2} \sum_{i=1}^{2} \left(\partial_\mu \phi_i \partial^\mu \phi_i - m^2 \phi_i^2 \right)}$$

Arbitrary convention (except for sign)

so that Hamiltonian is bounded below

Euler-Lagrange Equations: $\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) = \frac{\partial \mathcal{L}}{\partial \phi_i}$

$$\boxed{\partial_\mu \partial^\mu \phi_i = -m^2 \phi_i}$$

Conserved current: $\left.\frac{\partial \phi_1}{\partial \theta}\right|_{\theta=0} = \phi_2$

$$\left.\frac{\partial \phi_2}{\partial \theta}\right|_{\theta=0} = -\phi_1$$

Infinitesimal symmetry transformation:

$\phi_1 \to \phi_1 + \theta \phi_2, \qquad \phi_2 \to \phi_2 - \theta \phi_1$

$$S \to S + \int d^4x \sum_i \left(\frac{\partial \mathcal{L}}{\partial \phi_i} \delta\phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta(\partial_\mu \phi_i) \right)$$

$$= S + \int d^4x \sum_i \left[\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) \delta\phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\mu \delta\phi_i \right]$$

(using the E-L equations)

$$= S + \int d^4x \sum_i \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta\phi_i \right]$$

$$= S + \int d^4x \; \theta \sum_i \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \frac{\partial \phi_i}{\partial \theta}\Big|_{\theta=0} \right]$$

(using $\delta\phi_i = \theta \frac{\partial \phi_i}{\partial \theta}\Big|_{\theta=0}$)

But under the symmetry transformation $\mathcal{L} \to \mathcal{L}$, $S \to S$.

So $\partial_\mu J^\mu = 0$, where $\boxed{J^\mu = \sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \frac{\partial \phi_i}{\partial \theta}\Big|_{\theta=0}}$

More generally, a symmetry leaves S invariant,

infinitesimal symmetry parameter.

$\mathcal{L} \to \mathcal{L} + \theta \partial_\mu F^\mu$ for some $F^\mu(\phi_i(x), \partial_\mu \phi_i(x), x)$

w/o using the E-L equations.

Then $\boxed{J^\mu = \sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \frac{\partial \phi_i}{\partial \theta}\Big|_{\theta=0} - F^\mu}$

For our theory, $J^\mu = (\partial^\mu \phi_1)\phi_2 - (\partial^\mu \phi_2)\phi_1$

Interpretation of current conservation:

$$\partial_\mu J^\mu = 0$$

$\int d^3x$:
$$\int_V d^3x\, \partial_0 J^0 = -\int_V d^3x\, \partial_i J^i$$
$$= -\int_{\partial V} d^2x\, n^i J^i$$

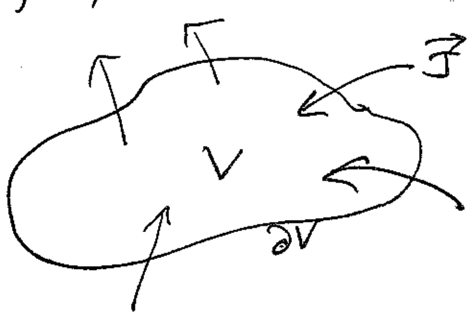
n^i = unit normal to boundary of V

Hence, $$\frac{d}{dt}\int_V d^3x\, J^0 = -\int_{\partial V} d^2x\, n^i J^i$$

Define <u>charge</u> $$\boxed{Q \equiv \int_V d^3x\, J^0}$$

Then $$\frac{dQ}{dt} = -\int_{\partial V} d^2x\, \hat{n}\cdot\vec{J}$$

i.e., rate of change of charge in a volume V is the flux of current into V



In our example, $$Q = \int d^3x\,(\phi_2\partial_0\phi_1 - \phi_1\partial_0\phi_2)$$

Complex scalar fields

It will often be convenient to combine pairs of real scalar fields into complex scalar fields:

$$\phi \equiv \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2), \qquad \phi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$$

$$(\partial_\mu \phi^*)(\partial^\mu \phi) = \frac{1}{2}(\partial_\mu \phi_1 - i\partial_\mu \phi_2)(\partial^\mu \phi_1 + i\partial^\mu \phi_2)$$

$$= \frac{1}{2}(\partial_\mu \phi_1)^2 + \frac{1}{2}(\partial_\mu \phi_2)^2$$

$$m^2 |\phi|^2 = \frac{1}{2} m^2 (\phi_1^2 + \phi_2^2)$$

Then the Lagrangian of our previous example can be rewritten as,

$$\mathcal{L} = |\partial_\mu \phi|^2 - m^2 |\phi|^2$$

Notice that the overall factor of $\frac{1}{2}$ has been absorbed in the normalization of ϕ.

Now let's be naive. Pretend we didn't know that ϕ and ϕ^* were related. Then we would treat ϕ and ϕ^* as independent fields, with

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) = \frac{\partial \mathcal{L}}{\partial \phi}, \qquad \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)}\right) = \frac{\partial \mathcal{L}}{\partial \phi^*}$$

For our theory, $\partial_\mu \partial^\mu \phi^* = -m^2 \phi^*$
$\partial_\mu \partial^\mu \phi = -m^2 \phi$

First of all, note that these Euler-Lagrange eqs are self-consistent — they are complex conjugates of one another.

In terms of ϕ_1 and ϕ_2, these become:

$$\partial_\mu \partial^\mu (\phi_1 - i\phi_2) = -m^2 (\phi_1 - i\phi_2)$$
$$\partial_\mu \partial^\mu (\phi_1 + i\phi_2) = -m^2 (\phi_1 + i\phi_2)$$

The real and imaginary parts are:

$$\partial_\mu \partial^\mu \phi_1 = -m^2 \phi_1$$
$$\partial_\mu \partial^\mu \phi_2 = -m^2 \phi_2$$

These were the equations of motion we derived earlier. We got the right answer by treating ϕ and ϕ^* as independent fields!

This will work quite generally. Why? Given an action $S(\phi, \phi^*)$, the equations of motion follow from $0 = \delta S = \int d^4x \Big[\Big(\partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Big) - \frac{\partial \mathcal{L}}{\partial \phi} \Big) \delta\phi$

$$+ \Big(\partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \Big) - \frac{\partial \mathcal{L}}{\partial \phi^*} \Big) \delta\phi^* \Big]$$

Treating $\delta\phi$ and $\delta\phi^*$ as independent we get

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial \mathcal{L}}{\partial\phi} = 0$$

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^*)}\right) - \frac{\partial \mathcal{L}}{\partial\phi^*} = 0 .$$

But we know that $\delta\phi$ and $\delta\phi^*$ are related. We can make purely real variations $\delta\phi = \delta\phi^*$, and we would have gotten

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\right) + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^*)}\right) - \frac{\partial \mathcal{L}}{\partial\phi} - \frac{\partial \mathcal{L}}{\partial\phi^*} = 0.$$

There's no funny business here. We used the fact that $\delta\phi$ and $\delta\phi^*$ are related, and considered a real variation of ϕ.

Similarly, we could have made a purely imaginary variation $\delta\phi = -\delta\phi^*$, and we would have gotten,

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^*)}\right) - \frac{\partial \mathcal{L}}{\partial\phi} + \frac{\partial \mathcal{L}}{\partial\phi^*} = 0.$$

But now adding or subtracting the equations of motion derived from the real and imaginary variations, we obtain the naive result obtained by treating $\delta\phi$ and $\delta\phi^*$ as independent!

In terms of the complex field ϕ, the symmetry

$$\begin{cases} \phi_1 \to \phi_1 \cos\theta + \phi_2 \sin\theta \\ \phi_2 \to \phi_2 \cos\theta - \phi_1 \sin\theta \end{cases}$$

becomes,

$$\phi \to \phi e^{i\theta} \qquad \left.\frac{\partial \phi}{\partial \theta}\right|_{\theta=0} = i\phi$$

$$\phi^* \to \phi^* e^{-i\theta} \qquad \left.\frac{\partial \phi^*}{\partial \theta}\right|_{\theta=0} = -i\phi^*$$

Treating ϕ and ϕ^* as independent fields, we would expect a conserved current,

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}(i\phi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)}(-i\phi^*)$$

$$= i\phi\,\partial^\mu \phi^* - i\phi^*\,\partial^\mu \phi$$

$$= 2\,\mathrm{Im}\left(\phi^* \partial^\mu \phi\right)$$

<u>Exercise</u>: Check that this is equivalent to the current obtained previously in terms of the real fields ϕ_1 and ϕ_2.

$$\boxed{\text{The lesson is that when dealing with complex fields, it is fair to treat the fields and their conjugates as independent.}}$$

The Energy-Momentum Tensor

As an application of Noether's theorem in field theory, consider a theory of a scalar field $\phi(x)$ invariant under spacetime translations.

$\phi(x) \to \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi(x)$ for infinitessimal a^μ.

Lagrangian density $\mathcal{L}(\phi, \partial_\mu \phi)$ transforms similarly:

$$\mathcal{L} \to \mathcal{L} + a^\mu \partial_\mu \mathcal{L} = \mathcal{L} + a^\nu \partial_\mu \left(\delta^\mu{}_\nu \mathcal{L} \right)$$

For each $\nu = 0,1,2,3$ there is a conserved current:

$$\boxed{T^\mu{}_\nu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu{}_\nu \mathcal{L}}$$ Energy-Momentum Tensor

Conserved charge associated w/ time translations:

$$\boxed{H = \int d^3x\, T^{00} = \int d^3x \left[\underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}}_{\Pi(x) \,=\, \text{Canonical Momentum}} \partial_0 \phi - \mathcal{L} \right]}$$ (Agrees w/ previously defined Hamiltonian)

Conserved charge associated w/ spatial translations:

$$P^i = \int d^3x\, T^{0i} = -\int d^3x\, T^0{}_i = -\int d^3x \left[\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial_i \phi \right], \text{ or}$$

$$\boxed{P^i = -\int d^3x\, \Pi(x)\, \partial_i \phi(x)}$$ Spatial Momentum