Review of Lagrangian Mechanics

Suppose a system is described by a set of N coordinates $q_i(t)$, $i=1,\ldots,N$.

Action Principle: There exists a functional of q_i, $\dot{q}_i$ and t, called the action, which is stationary about variations $\delta q_i(t)$ along a classical path.

Lagrangian $L(q_i, \dot{q}_i, t)$

Action $S = \int_{t_1}^{t_2} dt\, L(q_i, \dot{q}_i, t)$

Variation $q_i(t) \rightarrow q_i(t) + \delta q_i(t)$

$\delta q_i(t_1) = \delta q_i(t_2) = 0$ (Boundaries fixed)

$$\delta S = \int_{t_1}^{t_2} dt \sum_i \left[\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right]$$

$$= \int_{t_1}^{t_2} dt \sum_i \delta q_i \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right]$$

$\delta S = 0$ for arbitrary small variation $\delta q_i(t)$ along a classical path

$$\Rightarrow \boxed{\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0} \quad \text{Euler-Lagrange Eqs.}$$

Hamiltonian Formulation

Given a Lagrangian L, define the canonical momenta

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \rightsquigarrow \frac{dp_i}{dt} = \frac{\partial L}{\partial q_i} \quad \text{from Euler-Lagrange Eqs.}$$

Define the Hamiltonian $H(p_i, q_i, t) \equiv \sum_i p_i \dot{q}_i - L$

H must be written in terms of p's and q's, not the $\dot{q}$'s, and the p's and q's must be independent. (This is not always possible!)

Vary the coordinates and momenta:

$$dH = \sum_i \left(dp_i\, \dot{q}_i + \cancel{p_i\, d\dot{q}_i} - \frac{\partial L}{\partial q_i} dq_i - \cancel{\frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i} \right)$$

$$= \sum_i \left(\dot{q}_i\, dp_i - \dot{p}_i\, dq_i \right) \quad \text{using the Euler-Lagrange Eqs.}$$

$$= \sum_i \left(\frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i \right)$$

$$\rightsquigarrow \boxed{\frac{\partial H}{\partial p_i} = \dot{q}_i \ , \quad \frac{\partial H}{\partial q_i} = -\dot{p}_i} \quad \text{Hamilton's Eqs.}$$

$$\text{If } \frac{\partial L}{\partial t} = 0 \rightsquigarrow \frac{dH}{dt} = \sum_i \left(\frac{\partial H}{\partial p_i} \dot{p}_i + \frac{\partial H}{\partial q_i} \dot{q}_i \right) - \cancel{\frac{\partial L}{\partial t}}$$

$$= \sum_i \left(\dot{q}_i \dot{p}_i - \dot{p}_i \dot{q}_i \right) = 0$$

In that case H is called the energy, and is conserved.

Symmetries and Conservation Laws

Noether's Theorem: For every global symmetry parametrized by a continuous parameter there is a corresponding conservation law.

Under a variation of $q_i(t)$,

$$L \to L + \sum_i \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right)$$

$$= L + \sum_i \left(\dot{p}_i \, \delta q_i + p_i \, \delta \dot{q}_i \right) \quad \text{using the E-L Eqs.}$$

$$= L + \frac{d}{dt}\left(\sum_i p_i \, \delta q_i \right) \quad \text{using } \delta \dot{q}_i = \frac{d}{dt} \delta q_i$$

Now consider a class of variations parametrized by a continuous parameter λ: $\delta q_i(t, \lambda) = \lambda \left. \frac{\partial q_i}{\partial \lambda} \right|_{\lambda=0}$ as $\lambda \to 0$

Suppose under this class of transformations

$$L \to L + \lambda \frac{dF}{dt} \quad \text{for some } F(q_i, \dot{q}_i, t), \quad F(t_2) - F(t_1) = 0.$$

(not using the eqs. of motion)

Then the class of transformations $q_i(t) \to q_i(t, \lambda)$ is called a <u>symmetry</u>.

Hamilton's principle: $0 = \delta S = \int_{t_1}^{t_2} dt \; \delta L = \int_{t_1}^{t_2} dt \; \lambda \frac{dF}{dt}$

$$= \lambda \left(F(t_2) - F(t_1) \right) = 0$$

Such a symmetry transformation does not change the action or the equations of motion.

Around the eqs. of motion $L \to L + \frac{d}{dt}\left(\sum_i p_i \,\delta q_i(t,\lambda)\right)$

$$= L + \lambda \frac{d}{dt}\left(\sum_i p_i \frac{\partial q_i}{\partial \lambda}\right)\Big|_{\lambda=0} \text{ for } \lambda \ll 1$$

$$= L + \lambda \frac{dF}{dt} \text{ by assumption.}$$

$$\Rightarrow \boxed{\frac{d}{dt}\left(\sum_i p_i \frac{\partial q_i}{\partial \lambda}\Big|_{\lambda=0} - F(q_i, \dot{q}_i, t)\right) = 0}$$

The quantity $Q \equiv \sum_i p_i \frac{\partial q_i}{\partial \lambda}\Big|_{\lambda=0} - F(q_i, \dot{q}_i, t)$

is conserved.

<u>Example</u>: Space translation of point particles

Lagrangian $L = \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i^{\,2} - \sum_{ij} V_{ij}(|\vec{r}_i - \vec{r}_j|)$

$\vec{r}_i \to \vec{r}_i + \vec{a}\lambda$ takes $L \to L$ for any fixed $\vec{a}$.

$\frac{\partial \vec{r}_i(\lambda)}{\partial \lambda} = \vec{a}$, $F = 0$, $\vec{p}_i = \frac{\partial L}{\partial \dot{\vec{r}}_i} = m_i \dot{\vec{r}}_i$

$Q = \sum_i \vec{p}_i \cdot \vec{a}$ is conserved. Since this is true $\forall \vec{a}$,
$\sum_i \vec{p}_i$ is conserved.
Translation invariance $\longrightarrow$ Momentum Conservation

<u>Example</u>: Time translations $q_i(t) \to q_i(t+\lambda)$

$$\frac{\partial q_i(t,\lambda)}{\partial \lambda}\Big|_{\lambda=0} = \frac{d q_i(t)}{dt}$$

$$\delta L = \lambda \frac{dL}{dt} \quad \rightsquigarrow \quad F = L$$

$$Q = \sum_i P_i \dot{q}_i - L = H$$ is conserved.

Time translation invariance $\longrightarrow$ Energy conservation

Classical Field Theory

$q_i(t) \rightsquigarrow \phi_i(\vec{x}, t)$ infinite set of generalized coordinates

$t \rightsquigarrow t$

$i \rightsquigarrow i, \vec{x}$

$\sum_i \rightsquigarrow \sum_i \int d^3\vec{x}$

Lagrangian $L(q_i, \dot{q}_i, t) \rightsquigarrow$ Lagrangian density $\mathcal{L}(\phi_i, \partial_\mu \phi_i, x^\mu)$

$$L \equiv \int d^3x \, \mathcal{L}$$ integrated over a spacelike time slice.

We assume that $\mathcal{L}$ is local in space and time, and depends on at most first derivatives w.r.t. $t, \vec{x}$.

If $\mathcal{L}$ is a Lorentz scalar, then the Euler-Lagrange eqs will be Lorentz covariant.

$$\delta S = \sum_i \int \underbrace{dt\, d^3x}_{d^4x} \left[\frac{\partial \mathcal{L}}{\partial \phi_i} \delta\phi_i + \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}}_{\equiv \pi_i^\mu} \underbrace{\delta(\partial_\mu \phi_i)}_{=\partial_\mu \delta\phi_i} \right]$$

under arbitrary variations $\delta\phi_i$ such that

$\delta\phi_i(t_1, \vec{x}) = \delta\phi_i(t_2, \vec{x}) = 0.$

Integrate by parts $\rightsquigarrow \delta S = \sum_i \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \pi_i^\mu \right] \delta\phi_i = 0.$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \pi_i^\mu = \boxed{\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} = 0}$$

Euler-Lagrange Eqs.

The canonical momentum is $\Pi_i(t,\vec{x}) \equiv \Pi_i^0(t,\vec{x})$
(Don't think of Π^μ_i as a 4-vector generalization of the canonical momentum.)

Hamiltonian $H = \sum_i \int d^3x \underbrace{\left[\Pi_i \partial_0 \phi_i - \mathcal{L}\right]}$

Hamiltonian density $\mathcal{H} = \sum_i \left[\Pi_i \partial_0 \phi_i - \mathcal{L}\right]$

<u>Example</u>: Most general $\mathcal{L}$ satisfying:

(1) $\mathcal{L}$ is a Lorentz scalar.

(2) $\mathcal{L}$ built from one real scalar field $\phi = \phi^*$

(3) $\mathcal{L}$ quadratic in ϕ, $\partial_\mu \phi$

for linear eqs. of motion

$$\mathcal{L} = \frac{1}{2} a \left[\partial_\mu \phi \partial^\mu \phi + b \phi^2\right]$$

we can rescale $\phi \to \frac{\phi}{\sqrt{|a|}}$, $\mathcal{L} \to \pm \frac{1}{2}\left[\partial_\mu \phi \partial^\mu \phi + b\phi^2\right]$

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{\partial}{\partial(\partial_\mu \phi)}\left(\pm \frac{1}{2} \eta^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi \pm \frac{1}{2} b \phi^2\right)$$

Use $\frac{\partial(\partial_\alpha \phi)}{\partial(\partial_\mu \phi)} = \delta^\mu_\alpha$

$$\begin{aligned} \Pi^\mu &= \pm \frac{1}{2} \eta^{\alpha\beta}\left(\delta^\mu_\alpha (\partial_\beta \phi) + (\partial_\alpha \phi)\delta^\mu_\beta\right) \\ &= \pm \frac{1}{2}\left(\eta^{\mu\beta}(\partial_\beta \phi) + \eta^{\alpha\mu}(\partial_\alpha \phi)\right) \\ &= \pm \partial^\mu \phi \end{aligned}$$

$$\partial_\mu \Pi^\mu - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \implies \boxed{\pm\left(\partial_\mu \partial^\mu \phi - b\phi\right) = 0}$$

The Euler-Lagrange eqs. are the Klein-Gordon equation in this theory if we define $b = -m^2$.

$\Rightarrow$ Terms in $\mathcal{L}$ quadratic in the fields without derivative are related to masses.

$$\mathcal{H} = \pi^0 \partial_0 \phi - \mathcal{L} = \pm \left[\tfrac{1}{2} (\pi^0)^2 + \tfrac{1}{2} (\nabla \phi)^2 - \tfrac{1}{2} b \phi^2 \right]$$

$$= \pm \tfrac{1}{2} \left[(\pi^0)^2 + (\nabla \phi)^2 + m^2 \phi^2 \right]$$

Each term in brackets is positive definite and can be made arbitrarily large. For stability we need $m^2 > 0$, and the overall positive sign.

$$\Rightarrow \boxed{\mathcal{L} = \tfrac{1}{2} \left(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \right)}$$

This Lagrangian density defines the free scalar field.

Plane wave solutions: $\phi_{\vec{k}}(t, \vec{x}) = a_{\vec{k}} \, e^{-i(\omega t - \vec{k} \cdot \vec{x})} + \text{c.c.}$

$$(\partial_\mu \partial^\mu + m^2) \phi_{\vec{k}}(t, \vec{x}) = 0 \rightsquigarrow \boxed{\omega^2 = \vec{k}^2 + m^2}$$

m is called the mass of the free scalar field.

An arbitrary solution to the E-L eq. can be decomposed in plane waves.