Physical interpretation of Lorentz transformations:

Consider rotations: For a rotation about the x^3-axis by an angle θ we would write

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & & & \\ & \cos\theta & -\sin\theta & \\ & \sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

For small angles this becomes, $\begin{pmatrix} 1 & & & \\ & 1 & -\theta & \\ & \theta & 1 & \\ & & & 1 \end{pmatrix} + \mathcal{O}(\theta^2)$.

We can write the infinitessimal transformation matrix as $\delta^\mu{}_\nu + \omega^\mu{}_\nu$, where $\omega^\mu{}_\nu$ is the antisymmetric matrix $\begin{pmatrix} 0 & & & \\ & 0 & -\theta & \\ & \theta & 0 & \\ & & & 0 \end{pmatrix}$.

For a general infinitessimal rotation by θ about the $\hat{\theta}$ axis we would have for the spatial components $\omega^i{}_j$, $\omega^{ij} = -\omega^i{}_j = \omega^j{}_i = -\omega^{ji} = \sum_k \varepsilon^{ijk} \theta \hat{\theta}^k$.

For our rotation about x^3 we have $\omega^{12} = \theta$.

For a boost in the x^1-direction by velocity v,

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \longrightarrow \begin{pmatrix} \cosh w & \sinh w & & \\ \sinh w & \cosh w & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

where the boost parameter w satisfies

$$\cosh w = 1/\sqrt{1 - v^2/c^2}$$

$$\tanh w = v/c$$

For small $\frac{v}{c}$, the transformation matrix becomes,

$$\begin{pmatrix} 1 & v/c & & \\ v/c & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + \theta(v/c)^2$$

If we again write this as $\delta^\mu{}_\nu + w^\mu{}_\nu$, then in this example $w^0{}_1 = w^1{}_0 = w^{10} = -w^{01} = v/c$.

A general infinitessimal Lorentz transformation is specified by the <u>antisymmetric matrix $w^{\mu\nu}$</u>.

Count # parameters in $w^{\mu\nu}$: $\frac{4 \cdot 3}{2} = 6$

$= 3$ rotations $+ 3$ boosts ✓

We can understand the antisymmetry of $\omega_{\mu\nu}$ from the defining relation for Lorentz transformations:

$$\Lambda^{\nu}{}_{\mu} = \delta^{\nu}{}_{\mu} + \omega^{\nu}{}_{\mu} \quad , \quad \Lambda^{\mu}{}_{\nu}\, \eta_{\mu\beta}\, \Lambda^{\beta}{}_{\alpha} = \eta_{\nu\alpha}$$

$$(\delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu})\, \eta_{\mu\beta}\, (\delta^{\beta}{}_{\alpha} + \omega^{\beta}{}_{\alpha})$$

$$= \eta_{\nu\alpha} + \omega_{\alpha\nu} + \omega_{\nu\alpha} + \underbrace{\omega_{\beta\nu}\, \omega^{\beta}{}_{\alpha}}_{\mathcal{O}(\omega^2)}$$

(Note that we have been raising and lowering indices with $\eta_{\mu\beta}$.)

So, to linear order in ω, $\boxed{\omega_{\alpha\nu} = -\omega_{\nu\alpha}}$, as promised.

Now back to the Dirac equation. We will introduce a notation that makes the Dirac eqn appear Lorentz covariant, and then we will prove that it is.

$$i\hbar \frac{\partial \psi}{\partial t} + i\hbar c \vec{\alpha} \cdot \nabla \psi = \beta mc^2 \psi$$

Multiply by β, define four "γ-matrices":

$$\gamma^0 \equiv \beta$$
$$\gamma^i \equiv \beta \alpha^i \quad , \quad i = 1,2,3$$

Using $\beta^2 = 1$: $$\boxed{i\hbar c \gamma^\mu \frac{\partial}{\partial x^\mu} \psi = mc^2 \psi}$$

This looks covariant, but we should keep in mind that γ^μ are just 4 matrices — <u>they don't transform under Lorentz transformations.</u>

More notation: The Feynman slash, $\not{p} \equiv \gamma^\mu p_\mu$ for any 4-vector p^μ.

$$\rightsquigarrow \boxed{i\hbar c \not{\partial} \psi = mc^2 \psi}$$

Finally, in "natural units" $\hbar = c = 1$

$$\rightsquigarrow \boxed{(i\not{\partial} - m)\psi = 0}$$ The nice, compact Dirac eqn.

Two useful representations of the gamma matrices:

<u>Dirac basis</u>:
$$\gamma^0 = \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$
$$\gamma^i = \beta\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

<u>Weyl basis</u>:
$$\gamma^0 = \beta = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$
$$\gamma^i = \beta\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

<u>Exercise</u>: What is α^i in the Weyl basis?

<u>Properties of γ^μ</u>:
$$(\gamma^0)^2 = \beta^2 = \mathbb{1}$$
$$(\gamma^i)^2 = \beta\alpha^i\beta\alpha^i = -\beta^2(\alpha^i)^2 = -\mathbb{1}$$
$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0 \quad \text{if } \mu \neq \nu$$

Summary: $\boxed{\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}}$

We can rederive the Klein-Gordon eqn from the Dirac eqn using our new notation:

$$i\not{\partial}\psi = m\psi \qquad \text{Dirac eqn.}$$

$$-\not{\partial}\not{\partial}\psi = im\not{\partial}\psi = m^2\psi$$

$$-\gamma^\mu\partial_\mu\gamma^\nu\partial_\nu\psi = m^2\psi$$

$$= -\tfrac{1}{2}\{\gamma^\mu,\gamma^\nu\}\partial_\mu\partial_\nu\psi$$

$$= -\eta^{\mu\nu}\partial_\mu\partial_\nu\psi$$

$$= -\partial_\mu\partial^\mu\psi$$

Hence, $\boxed{-\partial_\mu\partial^\mu\psi = m^2\psi}$ Klein-Gordon eqn.

This is the manifestly covariant form of the K-G eqn. Both sides of the eqn. transform however ψ transforms.

To figure out how ψ transforms we can work backwards — start from the Dirac eqn and figure out what works.

Assume under a Lorentz transformation Λ,

$$\psi(x) \to S(\Lambda)\psi(\Lambda^{-1}x)$$

for some 4×4 matrix $S(\Lambda)$.

Suppose we now transform backwards by Λ^{-1}, so we get the sequence of transformations,

$$\psi(x) \to S(\Lambda)\psi(\Lambda^{-1}x) \to S(\Lambda^{-1})S(\Lambda\ \ )\psi(x).$$

But transforming by Λ and then Λ^{-1} should do nothing, so we get

$$\boxed{S(\Lambda^{-1}) = S^{-1}(\Lambda)}$$

Now transform the Dirac eqn:

$$(i\gamma^\mu \partial_\mu - m)\psi(x) \to \left[(i\gamma^\mu (\Lambda^{-1})^\nu{}_\mu \partial_\nu - m) S(\Lambda)\psi\right](\Lambda^{-1}x)$$

$$= 0$$

Multiply on the left by $S^{-1}(\Lambda)$:

$$i S^{-1}(\Lambda)\gamma^\mu S(\Lambda)\,(\Lambda^{-1})^\nu{}_\mu \partial_\nu \psi - m\psi = 0$$

This has the same form as the original Dirac eqn iff

$$\boxed{S^{-1}\gamma^\mu S (\Lambda^{-1})^\nu{}_\mu = \gamma^\nu}$$

Given an infinitessimal Lorentz transformation specified by the antisymmetric matrix $\omega^{\mu\nu}$, expand $S(\Lambda)$ in ω:

$$S \approx 1 - \frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}$$

$$S^{-1} \approx 1 + \frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}$$

for some set of matrices $\sigma_{\mu\nu}$, antisymmetric in $\mu \leftrightarrow \nu$.

$$S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu$$

$$\rightsquigarrow \gamma^\mu + \frac{i}{4}\sigma_{\alpha\beta}\omega^{\alpha\beta}\gamma^\mu - \frac{i}{4}\gamma^\mu\sigma_{\alpha\beta}\omega^{\alpha\beta} + \mathcal{O}(\omega^2)$$

$$= \gamma^\mu + \omega^\mu{}_\nu \gamma^\nu$$

So we need to find a set of matrices $\sigma_{\alpha\beta}$ satisfying,

$$\frac{i}{4}\sigma_{\alpha\beta}\omega^{\alpha\beta}\gamma^\mu - \frac{i}{4}\gamma^\mu\sigma_{\alpha\beta}\omega^{\alpha\beta} = \omega^\mu{}_\nu \gamma^\nu$$

The answer; $\sigma^{\alpha\beta} = \frac{i}{2}[\gamma^\alpha, \gamma^\beta]$

<u>Exercise</u>: Use the antisymmetry of $\omega^{\alpha\beta}$ and $\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta}$ to show that this $\sigma^{\alpha\beta}$ works.

We have found that the Dirac spinor transforms under infinitessimal Lorentz transformations by the matrix,

$$\boxed{S = \mathbb{1} + \frac{1}{8}[\gamma^\mu, \gamma^\nu]\omega_{\mu\nu}}$$

In the Dirac representation of the γ-matrices:

$$\sigma^{0i} = i\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad \sigma^{ij} = \varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

Given the infinitessimal Lorentz transformations in the Dirac spinor representation, we can build finite Lorentz transformations.

Recall how this works for rotations:

Rotate by R_1, then R_2:

$$\psi \to S(R_2)S(R_1)\psi = S(R_2 R_1)\psi$$

Rotations are generated by the angular momentum matrices J_i, $[J_i, J_j] = i\epsilon_{ijk} J_k$.

$$S(R) = \exp\left(-i\theta^i J^i\right)$$ rotates about $\hat{\theta}$ by θ.

Any matrices satisfying the algebra $[J_i, J_j] = i\epsilon_{ijk} J_k$ form a representation.

spin-½ representation: $J^i = \frac{\sigma^i}{2}$.

Lorentz transformations:

$$\psi \to S(\Lambda_2)S(\Lambda_1)\psi = S(\Lambda_2\Lambda_1)\psi$$

The matrices $\frac{1}{4}\sigma_{\mu\nu}$ generate Lorentz transformations in the Dirac spinor representation.

$$\rightsquigarrow \boxed{S(\Lambda) = \exp\left(-\frac{i}{4}\sigma_{\mu\nu}\,\omega^{\mu\nu}\right)}$$

The Lorentz transformations are parametrized by the antisymmetric matrix $\omega^{\mu\nu}$.

(To compare with Peskin Ch. 3.2, $S^{\mu\nu} = \frac{1}{2}\sigma^{\mu\nu}$.)

Compare with rotation of 2-component Pauli spinor in nonrelativistic QM:

$$u(\vec{x}) \to \exp\left(\frac{i}{2}\theta\,\hat{n}\cdot\vec{\sigma}\right)u(R^{-1}\vec{x})$$

Spatial rotations: σ_{ij} Hermitian $\to$ $S(\text{rotation}) =$ unitary.

But for boosts, σ_{0i} is not Hermitian:

$$S = \exp\left(-\frac{i}{2}\omega\,\sigma_{01}\right) = \exp\left(-\frac{\omega}{2}\alpha_1\right) = S^{+} \neq S^{-1}$$

So the transformation matrices for boosts are nonunitary.
However, $\boxed{S^{-1} = \gamma_0 S^{+} \gamma_0}$ is valid for both boosts & rot'ns.

Now consider the Lorentz transf of $\psi^{+}\psi$:

$$\psi^{+}\psi \to \psi^{+}S^{+}S\psi \neq \psi^{+}\psi$$

On the other hand,

$$\psi^{+}\gamma^0\psi \to \psi^{+}S^{+}\gamma^0 S\psi = \psi^{+}\overbrace{\gamma^0\gamma^0}^{1}S^{+}\gamma^0 S\psi$$

$$= \psi^{+}\gamma^0 S^{-1}S\psi = \psi^{+}\gamma^0\psi$$

So $\psi^{+}\gamma^0\psi$ is Lorentz invariant.

How about $\psi^\dagger \gamma^0 \gamma^\mu \psi$?

$$\psi^\dagger \gamma^0 \gamma^\mu \psi \to \psi^\dagger S^\dagger \gamma^0 \gamma^\mu S \psi$$
$$= \psi^\dagger \gamma^0 \gamma^0 S^\dagger \gamma^0 \gamma^\mu S \psi$$
$$= \psi^\dagger \gamma^0 S^{-1} \gamma^\mu S \psi$$
$$= \psi^\dagger \gamma^0 \Lambda^\mu{}_\nu \gamma^\nu \psi$$
$$= \Lambda^\mu{}_\nu \; \psi^\dagger \gamma^0 \gamma^\nu \psi$$

Hence, $\psi^\dagger \gamma^0 \gamma^\mu \psi$ transforms as a 4-vector.

The combination $\psi^\dagger \gamma^0$ appears so often it is given its own notation and a name:

$$\boxed{\bar{\psi} \equiv \psi^\dagger \gamma^0} \quad \text{Dirac adjoint.}$$

We can also define the Dirac adjoint of an operator:

$$\boxed{\bar{A} = \gamma^0 A^\dagger \gamma^0}$$

It is defined so that $\bar{\varphi} \bar{A} \psi = (\bar{\psi} A \varphi)^*$, like the ordinary adjoint.