Quantization of the Dirac Spinor Field - Part 2

We are now ready to attempt to canonically quantize the theory of the free Dirac spinor field.

The field satisfies the Dirac equation, and is decomposed as

$$\psi(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\left(e^{-ik\cdot x}\sum_{r=1,2} a_k^r u^r(k) + e^{ik\cdot x}\sum_{r=1,2} b_k^{r} v^r(k)\right).$$

The canonical momentum conjugate to ψ is

$$\Pi_\psi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = \pm i\psi^\dagger$$

To determine the commutation relations for the operators a_k^r, $a_k^{r\dagger}$, b_k^r and $b_k^{r\dagger}$, we impose the ETCR's on ψ and Π_ψ:

$$[\psi_\alpha(\vec{x},t), \psi_\beta(\vec{y},t)] = 0$$

$$[\Pi_{\psi_\alpha}(\vec{x},t), \Pi_{\psi_\beta}(\vec{y},t)] = 0$$

$$[\psi_\alpha(\vec{x},t), \Pi_{\psi_\beta}(\vec{y},t)] = i\,\delta^3(\vec{x}-\vec{y})\,\delta_{\alpha\beta}$$

where α and $\beta = 1,2,3,4$ are Dirac spinor indices.

$$\pm\left[\psi(\vec{x},t), i\psi^\dagger(\vec{y},t)\right] = \pm i\int\frac{d^3k\, d^3k'}{(2\pi)^6\sqrt{2\omega_k 2\omega_{k'}}}\Big(e^{-i(k\cdot x - k'\cdot y)}\sum_{r,s}[a_k^r, a_{k'}^{s\dagger}]u^r(k)u^s(k')^\dagger$$

$$+ e^{i(k\cdot x - k'\cdot y)}\sum_{r,s}[b_k^r, b_{k'}^{s\dagger}]v^r(k)v^s(k')^\dagger$$

$$+ e^{-i(k\cdot x + k'\cdot y)}\sum_{r,s}[a_k^r, b_{k'}^{s\dagger}]u^r(k)v^s(k')^\dagger$$

$$+ e^{i(k\cdot x + k'\cdot y)}\sum_{r,s}[b_k^r, a_{k'}^{s\dagger}]v^r(k)u^s(k')^\dagger\Big)$$

If
$$\boxed{\begin{aligned}&[a_k^r, b_{k'}^{s\dagger}] = [b_k^r, a_{k'}^{s\dagger}] = 0\\ &[a_k^r, a_{k'}^{s\dagger}] = [b_k^r, b_{k'}^{s\dagger}] = \pm(2\pi)^3\delta^3(\vec{k}-\vec{k}')\delta^{rs}\end{aligned}}$$

Then $[\psi(\vec{x},t), \pm i\psi^\dagger(\vec{y},t)]$

$$= i\int\frac{d^3k}{(2\pi)^3 2\omega_k}\Big(\sum_r u^r(k)\overbrace{\bar{u}^r(k)\gamma^0}^{u^r(k)^\dagger} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}$$

$$+ \sum_r v^r(k)\bar{v}^r(k)\gamma^0 e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}\Big)$$

$$= i\int\frac{d^3k}{(2\pi)^3 2\omega_k}\Big((\not{k}+m)\gamma^0 e^{i\vec{k}\cdot(\vec{x}-\vec{y})} + (\not{k}-m)e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}\Big)$$

$$= i\int\frac{d^3k}{(2\pi)^3 2\omega_k}\Big((\omega_k\gamma^0 + k_i\gamma^i + m) + (\omega_k\gamma^0 - k_i\gamma^i - m)\Big)\gamma^0 e^{i\vec{k}\cdot(\vec{x}-\vec{y})}$$

taking $k_i \to -k_i$

$$= i\int\frac{d^3k}{(2\pi)^3 2\omega_k}\cdot 2\omega_k\, e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\,\mathbb{1}$$

$$= i\,\delta^3(\vec{x}-\vec{y})\,\mathbb{1}, \text{ as desired.}$$

Similarly, if $\boxed{[a_k^r, a_{k'}^s] = [b_k^r, b_{k'}^s] = [a_k^r, b_{k'}^s] = 0}$

then $[\psi(x), \psi(y)]\big|_{x^0=y^0} = 0.$

And if $\boxed{[a_k^{r+}, a_{k'}^{s+}] = [b_k^{r+}, b_{k'}^{s+}] = [a_k^{r+}, b_{k'}^{s+}] = 0}$

then $[\psi(x)^+, \psi(y)^+]\big|_{x^0=y^0} = 0.$

Next, we calculate the Hamiltonian in terms of the creation and annihilation operators.

<u>Dirac Field Hamiltonian</u>: $H = \pm\int d^3x\, \overline{\psi}(-i\gamma^i\partial_i + m)\psi$

$$= \pm\int d^3x\, \overline{\psi}(i\gamma^0\partial_0)\psi$$

by the Dirac equation.

$$H = \pm\int d^3x \frac{d^3k\, d^3k'}{(2\pi)^6\sqrt{2\omega_k 2\omega_{k'}}}\Big(e^{ix\cdot(k-k')}(-i\omega_{k'})\sum_{r,s} a_k^{r+} a_{k'}^s\, \bar{u}^r(k)\, i\gamma^0 u^s(k')$$

$$+ e^{-ix\cdot(k-k')}(i\omega_{k'})\sum_{r,s} b_k^{r+} b_{k'}^s\, \bar{v}^r(k)\, i\gamma^0 v^s(k')$$

$$+ e^{ix\cdot(k+k')}(i\omega_{k'})\sum_{r,s} a_k^{r+} b_{k'}^s\, \bar{u}^r(k)\, i\gamma^0 v^s(k')$$

$$+ e^{-ix\cdot(k+k')}(-i\omega_{k'})\sum_{r,s} b_k^{r+} a_{k'}^s\, \bar{v}^r(k)\, i\gamma^0 u^s(k')\Big)$$

The $\int d^3x$ integrals give delta-functions as usual. The delta functions allow us to do the $\int d^3k'$ integrals.

Then, $$H = \pm \int \frac{d^3k}{(2\pi)^3 2\omega_k} \omega_k \sum_{rs} \Big(a_{\vec{k}}^{r\dagger} a_{\vec{k}}^{s} \bar{u}^r(\vec{k}) \gamma^0 u^s(\vec{k})$$

$$- b_{\vec{k}}^{r\dagger} b_{\vec{k}}^{s} \bar{v}^r(\vec{k}) \gamma^0 v^s(\vec{k})$$

$$- a_{\vec{k}}^{r\dagger} b_{-\vec{k}}^{s} \bar{u}^r(\vec{k}) \gamma^0 v^s(-\vec{k}) e^{2ix^0\omega_k}$$

$$+ b_{\vec{k}}^{r\dagger} a_{-\vec{k}}^{s} \bar{v}^r(\vec{k}) \gamma^0 u^s(-\vec{k}) e^{-2ix^0\omega_k} \Big)$$

To calculate expressions like $\bar{u}^r(k)\gamma^0 u^s(k)$ we can use the fact that $\bar{u}\gamma^0 u$ is the time component of a 4-vector $\bar{u}\gamma^\mu u$, and boost the expression from the rest frame.

In the rest frame, the Dirac equation gives

$$\left.\begin{aligned} \gamma^0 u &= u \\ \gamma^0 v &= -v \end{aligned}\right. \quad \text{(Rest frame)}$$

By the orthogonality relations,

$$\left.\begin{aligned} \bar{u}^r \gamma^0 u^s &= \bar{u}^r u^s = 2m\,\delta^{rs} \\ \bar{v}^r \gamma^0 v^s &= -\bar{v}^r v^s = 2m\,\delta^{rs} \\ \bar{u}^r \gamma^0 v^s &= \bar{v}^r \gamma^0 u^s = 0 \end{aligned}\right\} \text{ Rest frame}$$

Boosting to a general frame,

$$\boxed{\begin{aligned} \bar{u}^r(k)\gamma^0 u^s(k) &= 2\omega_k\,\delta^{rs} \\ \bar{v}^r(k)\gamma^0 v^s(k) &= 2\omega_k\,\delta^{rs} \\ \bar{u}^r(k)\gamma^0 v^s(k) &= 0 \\ \bar{v}^r(k)\gamma^0 u^s(k) &= 0 \end{aligned}}$$

Exercise: Using the solutions to the Dirac equation it is also the case that:

$$\boxed{\bar{v}^r(\vec{k})\gamma^0 u^s(-\vec{k}) = \bar{u}^r(\vec{k})\gamma^0 v^s(-\vec{k}) = 0}$$

Using these relations, the Hamiltonian becomes:

$$H = \pm\int \frac{d^3k}{(2\pi)^3 \cdot 2}\, 2\omega_k \sum_r \left(a_k^{r\dagger} a_k^r - b_k^{r\dagger} b_k^r\right)$$

$$= \pm\int \frac{d^3k}{(2\pi)^3}\, \omega_k \sum_r \left(a_k^{r\dagger} a_k^r - b_k^{r\dagger} b_k^r\right)$$

If we define the vacuum such that $a_k^r|0\rangle = b_k^r|0\rangle = 0$, then:

The energy of the state $b_{k_1}^\dagger \cdots b_{k_n}^\dagger|0\rangle$ is $\mp\sum_{i=1}^n \omega_{k_i}$.

The energy of the state $a_{k_1}^\dagger \cdots a_{k_n}^\dagger|0\rangle$ is $\pm\sum_{i=1}^n \omega_{k_i}$.

The particles and antiparticles carry opposite sign energy, so the Hamiltonian is unbounded below!

Neither choice of sign in the Lagrangian, nor a redefinition of the vacuum by say, $b_k^{r\dagger}|0\rangle = 0$, will cure this problem.

$\Rightarrow$ Equal-time commutation relations for Dirac spinor fields leads to disaster.

The solution to the problem lies in the <u>statistics</u> of the particles created by $a_k^{r\dagger}$ and $b_k^{r\dagger}$. We have inadvertently tried to make a theory of bosonic electrons, and failed.

What makes these particles bosons? The Hilbert space of states is constructed as in the scalar field, and we can choose the same normalization, i.e.

$$|\vec{k}, r\rangle = \sqrt{2\omega_k}\; a_{\vec{k}}^{r\dagger}|0\rangle \quad, \text{etc.}$$

Consider the two-electron state,

$$|k_1, r; k_2, s\rangle = \sqrt{2\omega_{k_1}}\sqrt{2\omega_{k_2}}\; a_{\vec{k}_1}^{r\dagger} a_{\vec{k}_2}^{s\dagger}|0\rangle$$

Switching the labels $k_1 \leftrightarrow k_2$ and $r \leftrightarrow s$, we have

$$|k_2, s; k_1, r\rangle = \sqrt{2\omega_{k_1}}\sqrt{2\omega_{k_2}}\; a_{\vec{k}_2}^{s\dagger} a_{\vec{k}_1}^{r\dagger}|0\rangle$$

$$= |k_1, r; k_2, s\rangle \quad \text{because } [a_{\vec{k}_1}^{r\dagger}, a_{\vec{k}_2}^{s\dagger}] = 0.$$

Hence, these multiparticle states are <u>symmetric</u> under exchange, i.e. <u>bosons</u>.

If we instead had had $\{a_{\vec{k}}^{r\dagger}, a_{\vec{k}'}^{s\dagger}\} = 0$, then we would have obtained $|k_1, r; k_2, s\rangle = -|k_2, s; k_1, r\rangle$, as appropriate for fermions.

For the Dirac spinor field, we must assume <u>equal-time anticommutation relations</u> to obtain a description of fermionic spin-1/2 particles with positive energy:

$$\{\psi_\alpha(x), i\psi_\beta^\dagger(y)\}\Big|_{x^0=y^0} = i\,\delta^3(\vec{x}-\vec{y})\,\delta_{\alpha\beta}$$

$$\{\psi_\alpha(x), \psi_\beta(y)\}\Big|_{x^0=y^0} = 0$$

$$\{\psi_\alpha(x)^\dagger, \psi_\beta(y)^\dagger\}\Big|_{x^0=y^0} = 0$$

Then we would deduce,

$$\boxed{\{a_{\vec{k}}^r, a_{\vec{k}'}^{s\dagger}\} = \{b_{\vec{k}}^r, b_{\vec{k}'}^{s\dagger}\} = (2\pi)^3\,\delta^3(\vec{k}-\vec{k}')\,\delta^{rs}}$$

All other anticommutators vanish.

Now consider the Hamiltonian. We still have

$$H = \pm\int\frac{d^3k}{(2\pi)^3}\,\omega_k \sum_r \left(a_k^{r\dagger} a_k^r - b_k^{r\dagger} b_k^r\right)$$

Suppose we redefine $b_k^r \leftrightarrow b_k^{r\dagger}$ while maintaining the conditions defining the vacuum $a_k^r|0\rangle = b_k^r|0\rangle = 0$ (for the newly redefined b_k^r).

In other words, we write the field as note.

$$\boxed{\psi(x) = \int\frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\left(e^{-ik\cdot x}\sum_r a_k^r u^r(k) + e^{ik\cdot x}\sum_r b_k^{r\dagger} v^r(k)\right)}$$

The Hamiltonian is now

$$\boxed{H = \int \frac{d^3k}{(2\pi)^3} \omega_k \sum_r \left(a_k^{r\dagger} a_k^r - b_k^r b_k^{r\dagger} \right)}$$

$$= \int \frac{d^3k}{(2\pi)^3} \omega_k \sum_r \left(a_k^{r\dagger} a_k^r + b_k^{r\dagger} b_k^r \right) - \int d^3k \, \omega_k \cdot \underbrace{2}_{\text{from } \sum_r} \delta^3(\vec{0})$$

There's an infinite constant, but we're used to that by now. We can redefine the Hamiltonian by removing the constant term, or we can (equivalently) normal order.

Normal Ordering for Fermions

In order to take into account the minus signs from anticommuting fermionic creation or annihilation operators, we define normal ordering for fermions as follows:

$$:a_k^{r\dagger} a_{k'}^s: = a_k^{r\dagger} a_{k'}^s$$

$$:a_k^r a_{k'}^{s\dagger}: = -a_{k'}^{s\dagger} a_k^r$$

(note the minus sign)

and similarly exchanging either or both a's with b's.

More generally, normal ordering for fermions moves all $a_k^\dagger$'s and $b_k^\dagger$'s to the left of a_k's and b_k's, with a factor of (-1) for each permutation of operators required to move them to the appropriate order.

The normal-ordered Hamiltonian is

$$:H: = \int \frac{d^3k}{(2\pi)^3} \omega_k \sum_r \left(a_k^{r\dagger} a_k^r + b_k^{r\dagger} b_k^r \right)$$

From now on we'll just call this H.
Just as for scalars, we obtain:

$$\boxed{\begin{aligned} [H, a_k^{r\dagger}] &= \omega_k a_k^{r\dagger}, & [H, a_k^r] &= -\omega_k a_k^r \\ [H, b_k^{r\dagger}] &= \omega_k b_k^{r\dagger}, & [H, b_k^r] &= -\omega_k b_k^r \end{aligned}}$$

Note that we get commutators here because H is quadratic in fermionic operators.

$$\left([AB, C] = A\{B,C\} - \{A,C\}B \right)$$

With the vacuum defined by $a_k^r |0\rangle = b_k^r |0\rangle = 0$, the energy of the state $a_{k_1}^{r_1\dagger} \cdots a_{k_n}^{r_n\dagger} b_{k_{n+1}}^{r_{n+1}\dagger} \cdots b_{k_{m+n}}^{r_{m+n}\dagger} |0\rangle$ is,

$$H a_{k_1}^{r_1\dagger} \cdots b_{k_{m+n}}^{r_{m+n}\dagger} |0\rangle = \left(\sum_{i=1}^{m+n} \omega_{k_i} \right) a_{k_1}^{r_1\dagger} \cdots b_{k_{m+n}}^{r_{m+n}\dagger} |0\rangle.$$

One-particle states are normalized as before:

$$|\vec{k}, r; e^-\rangle = \sqrt{2\omega_k}\, a_{\vec{k}}^{r\dagger} |0\rangle \quad \leftarrow e^- \text{ created by } a_{\vec{k}}^{r\dagger}$$

$$|\vec{k}, r; e^+\rangle = \sqrt{2\omega_k}\, b_{\vec{k}}^{r\dagger} |0\rangle \quad \leftarrow e^+ \text{ created by } b_{\vec{k}}^{r\dagger}$$

$$\langle \vec{k}, r; e^- | \vec{k}', s; e^- \rangle = \langle \vec{k}, r; e^+ | \vec{k}', s; e^+ \rangle = 2\omega_k (2\pi)^3 \delta^3(\vec{k} - \vec{k}') \delta^{rs}$$

$$\langle \vec{k}, r; e^- | \vec{k}', s; e^+ \rangle = 0$$

Comment on Spin vs. Statistics

We have learned that a bosonic theory of spin-1/2 fields is inconsistent. We would have been led to similar inconsistencies if we had tried to quantize the scalar field as a fermion, i.e. w/ anticommutation relations. In relativistic QFT (in 4 dimensions) there is a deep connection between spin and statistics: integer spin fields must be bosons, and $\frac{1}{2}$-integer spin fields must be fermions.