What is Quantum Field Theory?

A <u>field</u> is a physical quantity associated to each point in spacetime.

Quantum Field Theory is concerned with quantization of dynamical systems of fields.

This course will focus on <u>relativistic</u> quantum systems. At energies $E \gtrsim mc^2$ relativity is important, and particles can be produced. For example,

$$p + p \to p + p + \pi^0$$
$$p + p \to p + p + p + \bar{p}$$

Hence, we are forced to consider systems of many particles.

Even at lower energies, many-particle processes contribute to observables. Consider perturbation theory in nonrelativistic quantum mechanics (NRQM).

$$H \to H + \delta V \qquad \delta E_0 = \langle 0|\delta V|0\rangle + \sum_n \frac{|\langle 0|\delta V|n\rangle|^2}{E_0 - E_n} + \cdots$$

High-precision effects include relativistic corrections of order $\left(\frac{v}{c}\right)^2$. Intermediate states with extra particles contribute corrections of order

Typical energies $\to \dfrac{E}{mc^2} \sim \dfrac{mv^2}{mc^2} = \left(\dfrac{v}{c}\right)^2$ — same as relativistic corrections, so can't be ignored.

mc^2 ← typical energy denominator.

Recall the motivation for the Schrödinger Eqn.

1923: Louis de Broglie — suggests material particles can act like waves.

$$\psi(\vec{x},t) = A\exp\left[i(\underset{\text{wave vector}}{\vec{k}}\cdot\vec{x} - \underset{\text{angular frequency}}{\omega} t)\right] \quad \text{for free particle}$$

$$\left.\begin{array}{l}\vec{p} = \hbar\vec{k} \\ E = \hbar\omega\end{array}\right\} \text{Einstein-de Broglie relations.}$$

(1925-1926: Heisenberg, Born, Jordan, Pauli — Matrix Quantum Mechanics)

1926: Schrödinger eqn.

$E = \dfrac{\vec{p}^{\,2}}{2m}$ together w/ Einstein-de Broglie relations

Consistent w/ plane wave solutions if.

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi$$

To include interactions, postulate the general relationship
$$\left.\begin{array}{l}E \to i\hbar\frac{\partial}{\partial t} \\ \vec{p} \to -i\hbar\nabla\end{array}\right\} \text{Consistent w/ discussion above.}$$

$$E = \frac{\vec{p}^{\,2}}{2m} + V(\vec{x}) \longrightarrow \boxed{i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(\vec{x})\psi}$$

- Nonrelativistic Schrödinger Eqn.

1926: Following the logic of the non-relativistic Schrödinger Eqn, Schrödinger, Klein, Gordon wrote down the Relativistic Schrödinger Eqn = Klein-Gordon Eqn

$$E^2 = \vec{p}^2 c^2 + m^2 c^4 \rightarrow \boxed{-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = -\hbar^2 c^2 \nabla^2 \psi + m^2 c^4 \psi}$$

Plane-Wave Solutions: $\psi(\vec{x}, t) = A \exp\left[i(\vec{k}\cdot\vec{x} - \omega t)\right] + B \exp\left[-i(\vec{k}\cdot\vec{x} - \omega t)\right]$

where $\boxed{\hbar^2 \omega^2 = \hbar^2 \vec{k}^2 c^2 + m^2 c^4}$

- Dispersion Relation
- relates frequency and wavevector.

Q: Can the relativistic wavefunction ψ have a probabilistic interpretation as in non-relativistic QM?

A: No.

Non-relativistic: $i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V(\vec{x})\psi$ (1)

$-i\hbar \frac{\partial \psi^*}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi^* + V(\vec{x})\psi^*$ (2)

Multiply (1) by ψ^*, (2) by ψ, and subtract:

$$i\hbar\left(\psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi^*}{\partial t}\right) = \frac{-\hbar^2}{2m}\left(\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^*\right)$$

$$i\hbar \frac{\partial}{\partial t} |\psi|^2 = -\frac{\hbar^2}{2m} \nabla \cdot \left(\psi^* \nabla \psi - \psi \nabla \psi^*\right)$$

$$\frac{\partial}{\partial t} |\psi|^2 + \frac{\hbar}{m} \nabla \cdot \mathrm{Im}\left(\psi^* \nabla \psi\right) = 0$$

- Resembles current conservation $\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$

$\rho = |\psi|^2$ = probability density

$\vec{J} = \frac{\hbar}{m} \mathrm{Im}\left(\psi^* \nabla \psi\right)$ = probability current

Integrate current conservation eqn over $\vec{x}$:

$$\int_V d^3x \frac{\partial \rho}{\partial t} = -\int_V d^3x \, \nabla \cdot \vec{J}$$
$$= -\int_{\partial V} d^2x \, \hat{n} \cdot \vec{J} = 0 \quad \text{as } V \to \mathbb{R}^3 \text{ (all space)}$$

$$\frac{d}{dt}\int d^3x \, \rho(\vec{x},t) = \boxed{\frac{d}{dt}\int d^3x \, |\psi|^2 = 0}$$

① Hence, the total probability is conserved. If $\int d^3x \, |\psi(\vec{x},t)|^2 = 1$ at some time t, then the same is true at all times.

② Also note that $|\psi|^2 \geq 0$.

Conditions ① and ② are the reason $|\psi|^2$ may be interpreted as a probability density.

Repeat for the relativistic wavefunction:

$$-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = -\hbar^2 c^2 \nabla^2 \psi + m^2 c^4 \psi \qquad (1')$$

$$-\hbar^2 \frac{\partial^2 \psi^*}{\partial t^2} = -\hbar^2 c^2 \nabla^2 \psi^* + m^2 c^4 \psi^* \qquad (2')$$

Multiply (1') by $i\psi^*$, (2') by $i\psi$, and subtract.

$$\boxed{i\frac{\partial}{\partial t}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right) = ic^2 \nabla \cdot \left(\psi^* \nabla \psi - \psi \nabla \psi^*\right)}$$

This is of the form $\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$, with

$$\rho = \frac{i\hbar}{2mc^2}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right) = \frac{-\hbar}{mc^2} \operatorname{Im}\left(\psi^* \frac{\partial \psi}{\partial t}\right)$$

$$\vec{J} = \frac{-i\hbar}{2m}\left(\psi^* \nabla \psi - \psi \nabla \psi^*\right) = \frac{\hbar}{m} \operatorname{Im}\left(\psi^* \nabla \psi\right)$$

(1') $\frac{d}{dt}\int d^3x\, \rho = \frac{d}{dt}\int d^3x\, \mathrm{Im}\left(\psi^* \frac{\partial \psi}{\partial t}\right) = 0$

The quantity $\int d^3x\, \mathrm{Im}\left(\psi^* \frac{\partial \psi}{\partial t}\right)$ is conserved,

but the density $\rho = \mathrm{Im}\left(\psi^* \frac{\partial \psi}{\partial t}\right)$ is not positive semidefinite.

→ The analogy of condition (2) for the nonrelativistic probability density is not satisfied.

Hence, ρ is <u>not</u> a probability density in this case.

Other difficulties w/ the Klein-Gordon eqn:

- Including electromagnetic coupling, predictions for fine structure of hydrogen spectrum disagreed w/ detailed measurements by Paschen.

- There are negative-energy solutions $E = \pm\sqrt{\vec{p}^{\,2}c^2 + m^2c^4}$
  → There is no ground state. What would prevent a state from decaying to successively lower energy states?

The Dirac Equation (1928)

Dirac reasoned that the essential difference between the relativistic and non-relativistic Schrödinger eqns is that the former is 2^{nd} order in time derivatives, while the latter is 1^{st} order.

Q: Can an eqn of the form $i\hbar \frac{\partial \psi}{\partial t} = H\psi$ be consistent with Lorentz invariance?

A: Yes, if there are enough ψ's and H is a matrix of differential operators.

Lorentz invariance relates spatial and time coordinates, so conjecture that H is linear in spatial derivatives.

$$\boxed{i\hbar \frac{\partial \psi}{\partial t} = -i\hbar c\left(\alpha^1 \frac{\partial \psi}{\partial x^1} + \alpha^2 \frac{\partial \psi}{\partial x^2} + \alpha^3 \frac{\partial \psi}{\partial x^3}\right) + \beta mc^2 \psi}$$

— Dirac Equation

If α^i were just c-numbers (complex numbers) then the Dirac eqn would not transform consistently under rotations of the coordinates.

What if α^i, β are $N \times N$ matrices, and ψ has N components?

$$i\hbar \frac{\partial \psi_j}{\partial t} = -i\hbar c \sum_{k=1}^{N}\left(\alpha^1 \frac{\partial}{\partial x^1} + \alpha^2 \frac{\partial}{\partial x^2} + \alpha^3 \frac{\partial}{\partial x^3}\right)_{jk} \psi_k + \sum_{k=1}^{N} \beta_{jk}\, mc^2 \psi_k$$

$$\equiv \sum_{k=1}^{N} H_{jk}\, \psi_k$$

Iterating the Dirac eqn,

$$-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = -\hbar^2 c^2 \sum_{k,l=1}^{3} \frac{\alpha^k \alpha^l + \alpha^l \alpha^k}{2} \frac{\partial^2 \psi}{\partial x^k \partial x^l}$$

$$+ \frac{\hbar m c^3}{i} \sum_{k=1}^{3} \left(\alpha^k \beta + \beta \alpha^k\right) \frac{\partial \psi}{\partial x^k}$$

$$+ \beta^2 m^2 c^4 \psi$$

To recover $E^2 = \vec{p}^{\,2} c^2 + m^2 c^4$ for plane wave solutions, we arrange that each component of ψ satisfy the Klein-Gordon eqn,

$$-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = -\hbar^2 c^2 \nabla^2 \psi + m^2 c^4 \psi$$

In that case, (1) $\boxed{\alpha^k \alpha^l + \alpha^l \alpha^k = 2\delta^{kl} \mathbb{1}_{N\times N}}$

Kronecker delta $= \begin{cases} 1 & \text{if } k=l \\ 0 & \text{if } k \neq l \end{cases}$

(2) $\boxed{\alpha^k \beta + \beta \alpha^k = 0}$

(3) $\boxed{\beta^2 = \mathbb{1}_{N\times N}}$

- It follows from (1) and (3) that $(\alpha^k)^2 = \beta^2 = \mathbb{1}_{N\times N}$

$\rightarrow$ $\boxed{\text{eigenvalues of } \alpha^k, \beta \text{ are } \pm 1}$

- $0 = (\alpha^k \beta + \beta \alpha^k)\beta = \alpha^k + \beta \alpha^k \beta$

$\mathrm{Tr}\, \alpha^k = -\mathrm{Tr}\, \beta \alpha^k \beta = -\mathrm{Tr}\, \beta^2 \alpha^k = -\mathrm{Tr}\, \alpha^k$

cyclicity of trace $\quad\quad \beta^2 = \mathbb{1}_{N\times N}$

$\rightarrow$ $\boxed{\mathrm{Tr}\, \alpha^k = 0}$

- Similarly, $0 = \alpha^k(\alpha^k \beta + \beta \alpha^k) = \beta + \alpha^k \beta \alpha^k$

$\rightarrow$ $\boxed{\mathrm{Tr}\, \beta = 0}$

• $\mathrm{Tr}\,\alpha^k = \Sigma$(eigenvalues of α^k) $= 0$
$\rightarrow$ # $+1, -1$ eigenvalues equal. Similarly for β
$\Rightarrow$ $\boxed{\alpha^k, \beta \text{ must be even-dimensional matrices.}}$

• H Hermitian $\rightarrow$ $\boxed{\alpha^k, \beta \text{ Hermitian matrices}}$

In summary, we need 4 mutually anticommuting, even-dimensional Hermitian matrices.

2×2 Hermitian matrices: spanned by 3 mutually anticommuting Pauli matrices σ^k and the unit matrix $\mathbb{1}$.
$\rightarrow$ 2×2 is out.

(However, note that if the mass $m=0$, then there is no matrix β in the Dirac eqn, and we would only need 3 matrices $\rightarrow$ the Pauli σ-matrices would work.)

What about 4×4 matrices?

<u>Dirac basis</u>: $\alpha^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}$ $\qquad \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$

Each entry is a 2×2 matrix

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

What about the probability density?

$$i\hbar \frac{\partial \psi}{\partial t} = -i\hbar c \sum_{k=1}^{3} \alpha^k \frac{\partial \psi}{\partial x^k} + \beta mc^2 \psi \qquad (1'')$$

$$-i\hbar \frac{\partial \psi^+}{\partial t} = i\hbar c \sum_{k=1}^{3} \frac{\partial \psi^+}{\partial x^k} \alpha^k + mc^2 \psi^+ \beta \qquad (2'')$$

Left-multiply (1″) by ψ^+, right-multiply (2″) by ψ, subtract:

$$i\hbar \left(\psi^+ \frac{\partial \psi}{\partial t} + \frac{\partial \psi^+}{\partial t} \psi \right) = -i\hbar c \sum_{k=1}^{3} \left(\psi^+ \alpha^k \frac{\partial \psi}{\partial x^k} + \frac{\partial \psi^+}{\partial x^k} \alpha^k \psi \right)$$

$$\boxed{\frac{\partial}{\partial t} (\psi^+ \psi) + c \sum_{k=1}^{3} \frac{\partial}{\partial x^k} (\psi^+ \alpha^k \psi) = 0}$$

This is of the form $\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$, with

$$\rho = \psi^+ \psi$$
$$J^k = c\, \psi^+ \alpha^k \psi \quad , \quad k = 1,2,3$$

Note that $\rho \geq 0$ and is conserved, so it is a candidate for a probability density.