Causality in Quantum Field Theory

A measurement at spacetime point x should not be able to influence a measurement at spacetime point y if x and y are spacelike separated $\left((x-y)^2 < 0\right)$. Otherwise the sense of causality would be reversed in a boosted frame in which the time ordering of x and y is reversed.

For causality to be maintained we expect fields to commute at spacelike separated points:

$$\boxed{[\phi(x), \phi(y)] = 0 \quad \text{if} \quad (x-y)^2 < 0}$$

In fact, this is a consequence of the equal-time commutation relations. If $(x-y)^2 < 0$ then there is a Lorentz transformation which takes x and y to the same time t. But $[\phi(t,\vec{x}), \phi(t,\vec{y})] = 0$, which implies that $[\phi(x), \phi(y)] = 0$ in the original frame (by a Lorentz transformation of the fields).

At generic spacetime points,

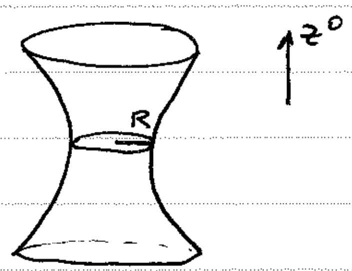
$$[\phi(x), \phi(y)] = \int \frac{d^3k \, d^3k'}{(2\pi)^6 \sqrt{2\omega_k 2\omega_{k'}}} \left[\left(a_k e^{-ik\cdot x} + a_k^\dagger e^{ik\cdot x}\right), \left(a_{k'} e^{-ik'\cdot y} + a_{k'}^\dagger e^{ik'\cdot y}\right)\right]$$

$$= \int \frac{d^3k \, d^3k'}{(2\pi)^6 \sqrt{2\omega_k 2\omega_{k'}}} (2\pi)^3 \delta^3(\vec{k} - \vec{k}') \left(e^{-i(k\cdot x - k'\cdot y)} - e^{i(k\cdot x - k'\cdot y)}\right)$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)}\right)$$

Recall that $(x-y)^2$ is Lorentz invariant, but Lorentz transformations take one point on the hyperboloid of constant $(x-y)^2$ to another. If $(x-y)^2 < 0$ then the hyperboloid has a single connected component. Define $z \equiv x-y$. $(z^0)^2 - \vec{z}^{\,2} = -R^2$

$$\Rightarrow \vec{z}^{\,2} = (z^0)^2 + R^2$$



The points $(z^0, \vec{z})$ and $-(z^0, \vec{z})$ are connected by a continuous Lorentz transformation along the hyperboloid.

Suppose the Lorentz transformation that does this is given by $\Lambda^\mu{}_\nu$, so that $\Lambda^\mu{}_\nu z^\nu = -z^\mu$. Then $k_\mu \Lambda^\mu{}_\nu z^\nu = -k \cdot z$. Now consider

$$\int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{ik\cdot z} = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \exp[i(\Lambda^\mu{}_\nu k_\mu) z^\nu]$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-ik\cdot z}$$

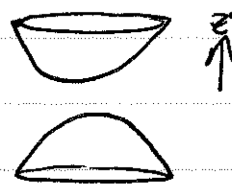
In the first equality we transformed $k_\mu \to \Lambda^\mu{}_\nu k_\mu$ and used the fact that the measure is Lorentz invariant. Hence, if $(x-y)^2 < 0$,

$$[\phi(x), \phi(y)] = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)} \right)$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(e^{-ik\cdot(x-y)} - e^{-ik\cdot(x-y)} \right)$$

$= 0$, as expected.

If $(x-y)^2 > 0$ then the hyperboloid $(x-y)^2 = R^2$ has two connected components, and there is no continuous Lorentz transformation that takes $(x-y)$ to $-(x-y)$.



For timelike separated x and y, $[\phi(x), \phi(y)]$ does not vanish.

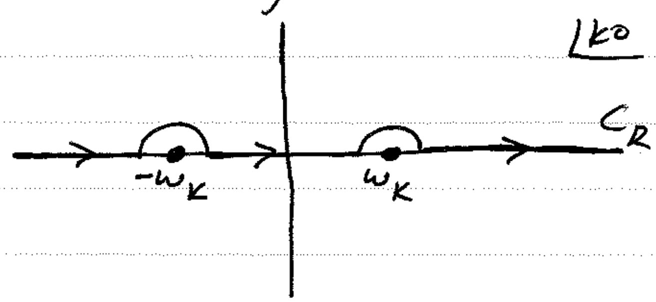
To evaluate the commutator more generally, take $\vec{k} \to -\vec{k}$ in the second integral:

$$[\phi(x), \phi(y)] = \int \frac{d^3k}{(2\pi)^3} \left(\left. \frac{e^{-ik\cdot(x-y)}}{2\omega_k} \right|_{k^0=\omega_k} - \left. \frac{e^{-ik\cdot(x-y)}}{2\omega_k} \right|_{k^0=-\omega_k} \right)$$

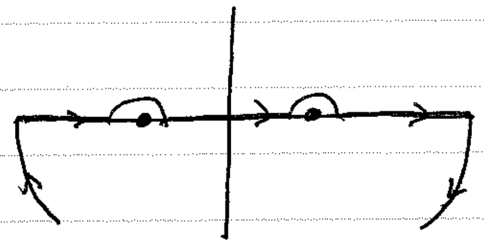
We can write the commutator in terms of Green's functions of the Klein-Gordon operator.

Define $$D_R(x-y) \equiv \int \frac{d^3k}{(2\pi)^3} \int_{C_R} \frac{dk^0}{2\pi} \frac{i\, e^{-ik\cdot(x-y)}}{k^2 - m^2}$$

where we now integrate k^0 over the contour C_R below:



Consider $D_R(x-y)$. If $x^0 > y^0$ then close the contour in the lower half plane.



$$\int \frac{dk^0}{2\pi} \frac{i e^{-ik^0(x^0-y^0)}}{(k^0)^2 - (\vec{k}^2 + m^2)}$$

$$= \frac{i}{2\pi}(-2\pi i)\left(\frac{e^{-i\omega_k(x^0-y^0)}}{-2\omega_k} + \frac{e^{+i\omega_k(x^0-y^0)}}{2\omega_k}\right)$$

by the residue theorem. Note that if $x^0 > y^0$,

$$\int \frac{d^3k}{(2\pi)^3} \int_{C_R} \frac{dk^0}{2\pi} \frac{i e^{-ik\cdot(x-y)}}{k^2 - m^2} = [\phi(x), \phi(y)]$$

If $x^0 < y^0$, close the contour C_R in the upper half plane, so that the contour does not enclose either of the poles. Then $D_R(x-y) = 0$.

To summarize our results so far,

$$\boxed{D_R(x-y) = \theta(x^0-y^0)\,[\phi(x), \phi(y)]}$$

Note that $\left(\partial_\mu \partial_x^\mu + m^2\right) D_R(x-y) = -i \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)}$,

or $$\boxed{\left(\partial_\mu \partial_x^\mu + m^2\right) D_R(x-y) = -i\, \delta^4(x-y)}$$

Hence, $i D_R(x-y)$ is a Green's function of the Klein-Gordon operator. Since $D_R(x-y)$ vanishes if $x^0 < y^0$, it is the <u>retarded</u> Green's function.

The causal behavior of the Green's function, i.e. retarded, advanced or otherwise, is determined by the contour chosen to define the function, namely how we handle the poles in the contour integral.

Closing the contour above both poles gave a retarded Green's function. Similarly, closing the contour below both poles would give an advanced Green's function.

Yet another contour gives the Feynman propagator.

The Feynman Propagator

We define a time-ordering operator T that orders products of fields so that fields at later times are to the left of fields at earlier times. For example,

$$T\big(\phi(x)\phi(y)\big) = \begin{cases} \phi(x)\phi(y) & \text{if } x^0 > y^0 \\ \phi(y)\phi(x) & \text{if } y^0 > x^0 \end{cases}$$

The Feynman propagator is defined by

$$\boxed{D_F(x-y) = \langle 0 | T\big(\phi(x)\phi(y)\big) | 0 \rangle}$$

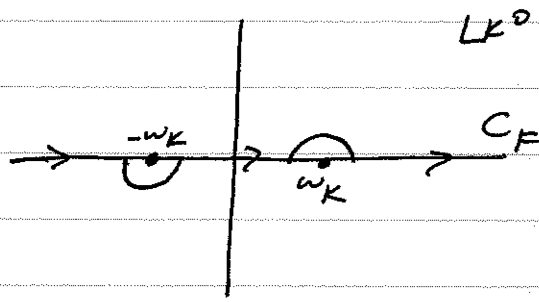
$\langle 0| T(\phi(x)\phi(y)) |0\rangle$

$$= \theta(x^0-y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y^0-x^0)\langle 0|\phi(y)\phi(x)|0\rangle$$

$$= \theta(x^0-y^0)\int\frac{d^3k}{(2\pi)^3 2\omega_k}e^{-ik\cdot(x-y)} + \theta(y^0-x^0)\int\frac{d^3k}{(2\pi)^3 2\omega_k}e^{ik\cdot(x-y)}$$

$$= \int_{C_F}\frac{d^4k}{(2\pi)^4}\frac{i\,e^{-ik\cdot(x-y)}}{k^2-m^2}$$

where the contour C_F for the k^0 integral is:



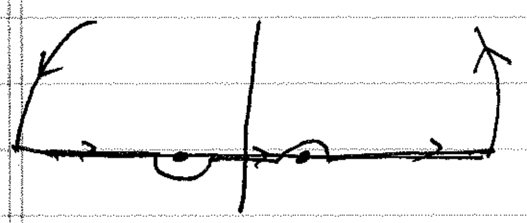
Check: If $x^0 > y^0$, close contour around lower half plane:

$$\int\frac{d^3k}{(2\pi)^3}\int\frac{dk^0}{(2\pi)}\frac{i\,e^{-ik\cdot(x-y)}}{(k^0)^2-\omega_k^2} = \int\frac{d^3k}{(2\pi)^3}\frac{(-2\pi i)}{2\pi}\frac{i\,e^{-ik\cdot(x-y)}}{2\omega_k}\Bigg|_{k^0=\omega_k}$$

$$= \int\frac{d^3k}{(2\pi)^3}\frac{e^{-ik\cdot(x-y)}}{2\omega_k}\Bigg|_{k^0=\omega_k}$$

$$= \langle 0|\phi(x)\phi(y)|0\rangle \qquad \checkmark$$

If $y^0 > x^0$ close contour around upper half plane:



$$\int \frac{d^3k}{(2\pi)^3} \int \frac{dk^0}{2\pi} \frac{i e^{-ik\cdot(x-y)}}{(k^0)^2 - \omega_k^2} = \int \frac{d^3k}{(2\pi)^3} \frac{2\pi i}{2\pi} \frac{i e^{-ik\cdot(x-y)}}{-2\omega_k}\Bigg|_{k^0=-\omega_k}$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik\cdot(x-y)}}{2\omega_k}\Bigg|_{k^0=\omega_k}$$

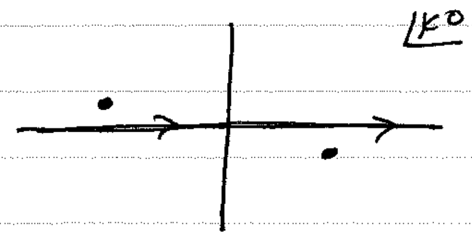
(taking $\vec{k} \to -\vec{k}$ in the integral)

$$= \langle 0 | \phi(y)\, \phi(x) | 0 \rangle$$

An easy way to remember where the poles go with respect to the contour is to write

$$\langle 0 | T(\phi(x)\,\phi(y)) | 0 \rangle = \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik\cdot(x-y)}}{k^2 - m^2 + i\epsilon}$$

where $\epsilon \to 0$ and we integrate k^0 over the real axis.



Note that the Feynman propagator is a Green's function for the Klein-Gordon operator!

$$\left((\partial_\mu \partial^\mu)_x + m^2\right) \langle 0| T(\phi(x)\phi(y))|0\rangle = -\int \frac{d^4k}{(2\pi)^4}\, i\, e^{-ik\cdot(x-y)}$$

$$= -i\,\delta^4(x-y)$$

Antiparticles and causality

Consider a free complex scalar field $\phi(x)$.

$$[\phi(x), \phi^\dagger(y)] = \int \frac{d^3k\, d^3k'}{(2\pi)^6\sqrt{2\omega_k\, 2\omega_{k'}}}\left([a_k, a^\dagger_{k'}]\, e^{-ik\cdot(x-y)} + [b^\dagger_k, b_{k'}]\, e^{ik\cdot(x-y)}\right)$$

$$= \int \frac{d^3k}{(2\pi)^3\, 2\omega_k}\left(e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)}\right)$$

This is the same as $[\phi(x), \phi(y)]$ for the real scalar field. The commutator vanishes if $(x-y)^2 < 0$ because the two terms in the integral cancel.

If $\phi(x)$ contained only its positive-frequency part, with coefficients a_k, then we would not have gotten the second term in the integral (which is required for causality). Hence, the antiparticles created by $b^\dagger_k$ are essential for causality in this theory.
For a real scalar field $\phi(x)$ the particle is its own antiparticle.

Note that

$$\langle 0|\phi(x)\phi^\dagger(y)|0\rangle = \left(\langle 0|\int\frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\,a_k e^{-ik\cdot x}\right)\left(\int\frac{d^3k'}{(2\pi)^3\sqrt{2\omega_{k'}}}\,a_{k'}^\dagger e^{ik'\cdot y}|0\rangle\right),$$

which we may interpret as the amplitude for a particle to propagate from y to x. Similarly,

$$\langle 0|\phi^\dagger(y)\phi(x)|0\rangle = \left(\langle 0|\int\frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\,b_k e^{-ik\cdot y}\right)\left(\int\frac{d^3k'}{(2\pi)^3\sqrt{2\omega_{k'}}}\,b_{k'}^\dagger e^{ik'\cdot x}|0\rangle\right),$$

which we may interpret as the amplitude for an antiparticle to propagate from x to y.

Outside the lightcone each of these amplitudes $\sim e^{-m|\vec{x}-\vec{y}|}$, but the two amplitudes cancel in the commutator.