The Dirac Equation: Coupling to Electromagnetism

To include the electromagnetic coupling to the electron, replace $\vec{p} \to \vec{p} - \frac{e}{c}\vec{A}$, $H \to H - e\phi$, where

$\vec{p}$ ← momentum H ← Hamiltonian

$\vec{A}$ = vector potential
ϕ = scalar potential
e = electric charge of electron (which is negative)

$$\left(\begin{array}{l} \vec{B} = \nabla \times \vec{A} \\ \vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \end{array} \right)$$

Write the Dirac wavefunction in terms of 2-component wavefunctions $\tilde{\varphi}$, $\tilde{\chi}$:

$$\psi = \begin{pmatrix} \tilde{\varphi} \\ \tilde{\chi} \end{pmatrix} \quad \text{(in the Dirac basis)}$$

In the Dirac basis the Dirac equation becomes

$$\boxed{i\hbar \frac{\partial}{\partial t}\begin{pmatrix} \tilde{\varphi} \\ \tilde{\chi} \end{pmatrix} = c\vec{\sigma}\cdot\left(\vec{p} - \frac{e}{c}\vec{A}\right)\begin{pmatrix} \tilde{\chi} \\ \tilde{\varphi} \end{pmatrix} + e\phi \begin{pmatrix} \tilde{\varphi} \\ \tilde{\chi} \end{pmatrix} + mc^2 \begin{pmatrix} \tilde{\varphi} \\ -\tilde{\chi} \end{pmatrix}}$$

In the nonrelativistic limit, $E \approx mc^2$ (for positive energy solutions). To focus on the interesting dynamics, factor out $e^{-imc^2t/\hbar}$ from the wavefunction:

Write $\begin{pmatrix} \tilde{\varphi} \\ \tilde{\chi} \end{pmatrix} = e^{-imc^2t/\hbar}\begin{pmatrix} \varphi \\ \chi \end{pmatrix}$

so that we expect φ, χ to be slowly varying functions of time for these solutions in the NR limit.

In terms of φ, χ the Dirac equation becomes

$$i\hbar \frac{\partial}{\partial t}\begin{pmatrix}\varphi\\ \chi\end{pmatrix} = c\vec{\sigma}\cdot\left(\vec{p}-\frac{e\vec{A}}{c}\right)\begin{pmatrix}\chi\\ \varphi\end{pmatrix} + e\phi\begin{pmatrix}\varphi\\ \chi\end{pmatrix} - 2mc^2\begin{pmatrix}0\\ \chi\end{pmatrix}$$

For small interaction energy, small kinetic energy, the second two components of the Dirac eqn give

$$\chi \approx \frac{1}{2mc}\,\vec{\sigma}\cdot\left(\vec{p}-\frac{e}{c}\vec{A}\right)\varphi$$

→ The components of χ are "small" compared to φ, reduced by $\sim v/c$

Inserting the approx. sol'n for χ into the top two components of the Dirac eqn,

$$\boxed{i\hbar\frac{\partial\varphi}{\partial t} = \frac{1}{2m}\vec{\sigma}\cdot\left(\vec{p}-\frac{e}{c}\vec{A}\right)\vec{\sigma}\cdot\left(\vec{p}-\frac{e}{c}\vec{A}\right)\varphi + e\phi\varphi}$$

To simplify the right-hand side, use identities for Pauli matrices:

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}$$

→ (1): $$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = 2\sum_{ij}\delta_{ij}\,a^i b^j - \sum_{ij}\sigma_j\sigma_i\,a^i b^j$$

$$[\sigma_i, \sigma_j] = 2i\sum_k \epsilon_{ijk}\,\sigma^k$$

→ (2): $$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = 2i\sum_{ijk} a^i b^j \sigma^k \epsilon_{ijk} + \sum_{ij} a^i b^j \sigma_j\sigma_i$$

Add the two expressions for $(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b})$:

$$(1)+(2) = 2(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = 2i\,\vec{\sigma}\cdot(\vec{a}\times\vec{b}) + 2\vec{a}\cdot\vec{b}$$

$$\Rightarrow \boxed{(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + i\,\vec{\sigma}\cdot(\vec{a}\times\vec{b})}$$

The Dirac Hamiltonian includes,

$$\frac{1}{2m}\,\vec{\sigma}\cdot\left(\vec{p}-\tfrac{e}{c}\vec{A}\right)\vec{\sigma}\cdot\left(\vec{p}-\tfrac{e}{c}\vec{A}\right)$$

$$= \frac{1}{2m}\left[\left(\vec{p}-\tfrac{e}{c}\vec{A}\right)\cdot\left(\vec{p}-\tfrac{e}{c}\vec{A}\right) + i\vec{\sigma}\cdot\left(\vec{p}-\tfrac{e}{c}\vec{A}\right)\times\left(\vec{p}-\tfrac{e}{c}\vec{A}\right)\right]$$

What is $\left(\vec{p}-\tfrac{e}{c}\vec{A}\right)\times\left(\vec{p}-\tfrac{e}{c}\vec{A}\right)$?

(Remember, these are operators so $\vec{V}\times\vec{V}$ does not necessarily vanish!)

$$\left(\vec{p}-\tfrac{e}{c}\vec{A}\right)\times\left(\vec{p}-\tfrac{e}{c}\vec{A}\right)$$

$$= \cancelto{0}{\vec{p}\times\vec{p}} - \frac{e}{c}\left(\vec{p}\times\vec{A} + \vec{A}\times\vec{p}\right) + \cancelto{0}{\frac{e^2}{c^2}\vec{A}\times\vec{A}}$$

$$= -\frac{e}{c}\frac{\hbar}{i}\nabla\times\vec{A} \qquad = i\frac{e\hbar}{c}\vec{B}$$

The Dirac equation then gives

$$\boxed{i\hbar\frac{\partial\psi}{\partial t} = \left[\frac{\left(\vec{p}-\frac{e}{c}\vec{A}\right)^2}{2m} - \frac{e\hbar}{2mc}\vec{\sigma}\cdot\vec{B} + e\phi\right]\psi}$$

– valid in the NR limit.

In a uniform weak magnetic field $\vec{B}$, $\vec{A} = \frac{1}{2}\vec{B}\times\vec{r}$, $\phi = 0$.
(*Exercise*: check $\vec{B} = \nabla\times\vec{A}$)

$$\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 = \left(\vec{p} - \frac{e}{2c}\vec{B}\times\vec{r}\right)\cdot\left(\vec{p} - \frac{e}{2c}\vec{B}\times\vec{r}\right)$$

$$= \vec{p}^{\,2} - \frac{e}{2c}\left((\vec{B}\times\vec{r})\cdot\vec{p} + \vec{p}\cdot(\vec{B}\times\vec{r})\right) + \underbrace{\mathcal{O}(\vec{B}^2)}_{\text{small}}$$

Use $\vec{a}\cdot\vec{b}\times\vec{c} = -(\vec{a}\times\vec{c})\cdot\vec{b}$ (being careful about operator ordering!)

$$\vec{p}\cdot(\vec{B}\times\vec{r}) = -(\vec{p}\times\vec{r})\cdot\vec{B} = \vec{L}\cdot\vec{B}$$

Similarly, $\vec{B}\times\vec{r}\cdot\vec{p} = \vec{r}\times\vec{p}\cdot\vec{B} = \vec{L}\cdot\vec{B}$

$$\rightarrow \left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 = \vec{p}^{\,2} - \frac{e}{c}\vec{L}\cdot\vec{B} + \mathcal{O}(\vec{B}^2)$$

$$\boxed{i\hbar\frac{\partial \psi}{\partial t} = \left[\frac{\vec{p}^{\,2}}{2m} - \frac{e}{2mc}\left(\vec{L} + 2\vec{S}\right)\cdot\vec{B}\right]\psi} \qquad (\star)$$

where $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$.

The classical spin-orbit interaction is given by the Hamiltonian $H_{\text{spin-orbit}} = -\vec{\mu}\cdot\vec{B}$.

The classical magnetic moment for an electron with speed v moving in a circle of radius r is

$$|\vec{\mu}| = \frac{ev}{2\pi r}\cdot\frac{\pi r^2}{c} = \frac{e}{2mc}|\vec{L}|$$

Comparing w/ the $\vec{S}\cdot\vec{B}$ term in $(\star)$, we find the prediction $|\vec{\mu}|/|\vec{S}| = \frac{e}{mc}$, or twice the classical expectation.

The gyromagnetic ratio g of the electron is defined by $|\vec{\mu}|/|\vec{S}| = g\, e/2mc$.

The Dirac equation predicts $\boxed{g=2}$, which (approximately) agrees with measurements of the Zeeman splitting of hydrogen.

Measurements by I. I. Rabi of the hyperfine structure of hydrogen and deuterium concluded that g differs from 2 by about $\frac{1}{10}$%.
The difference $g-2$ is called the <u>anomalous magnetic moment</u> of the electron. Using quantum electrodynamics it was calculated to 5 decimal places by J. Schwinger. You will do the same next semester in Phys 722.