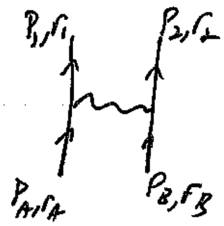
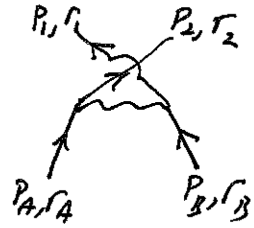
V9

Generic Features of Feynman Diagram Calculations

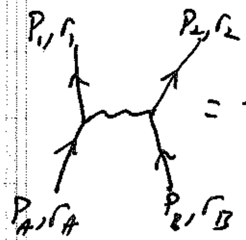
We have already calculated two Feynman diagrams that contribute to <u>electron-electron scattering</u>:



and



The first stands for



$$= -(-ie)^2 \frac{-i g_{\mu\nu}}{(p_2-p_B)^2+i\epsilon} \bar{u}^{r_2}(p_2)\gamma^\mu u^{r_B}(p_B)\, \bar{u}^{r_1}(p_1)\gamma^\nu u^{r_A}(p_A)$$

$$\times (2\pi)^4 \delta^4(p_1+p_2-p_A-p_B).$$

The d^4x integrals lead to momentum conservation at each vertex. There will always be an overall energy-momentum conserving δ-function left over.

The factors of $\bar{u}^r(p)$ came from $\langle 0 | a_p^r \bar{\psi}$, by anticommuting the $a_k^{s\dagger}$ in $\bar{\psi}$ past the a_p^r.

The factors of $u^r(p)$ came from $\psi a_p^{r\dagger} | 0 \rangle$ by anticommuting the a_k^s in ψ past the $a_p^{r\dagger}$.

The $ie\gamma^\mu$ and $ie\gamma^\nu$ came from the vertices $ie\bar{\psi}\gamma^\mu\psi$, $ie\bar{\psi}\gamma^\nu\psi$

The <u>photon propagator</u> $\frac{-i g_{\mu\nu}}{k^2+i\epsilon}$ came from the contraction $A_\mu(x) A_\nu(y)$.

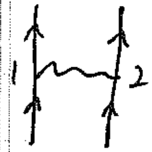
The minus sign in front was a result of normal-ordering of the fermions. There is no general rule for these minus signs — you have to keep track of them. Sometimes only relative minus signs between different Feynman diagrams are important, and these may be easy to identify.

Aside from the minus signs, all of the factors in the Feynman diagram calculation could have been obtained by some simple rules;

① Start at the tip of one of the external electron lines. There is a factor of $\bar{u}^r(p)$ for the external outgoing electron.
② Work against the arrow. When you reach a vertex you get a factor of $-ie\gamma^\mu$ if the photon line leaving the vertex has Lorentz index μ.
③ There is a factor of $u^r(p)$ for the external ingoing electron.
④ Repeat at each fermion line.
⑤ There is a factor of $\frac{-i g_{\mu\nu}}{k^2+i\epsilon}$ for each photon propagator connecting vertices labeled by μ and ν.

Let's generalize to other processes.
Consider <u>electron-positron scattering</u> to $O(e^2)$.

We need a different matrix element of the same Wick diagram as contributed to electron-electron scattering:



$$= \frac{(-ie)^2}{2!}\int d^4x_1\, d^4x_2 \; :\bar{\psi}\gamma^\mu\psi(x_1)\,\bar{\psi}\gamma^\nu\psi(x_2): \overbrace{A_\mu(x_1)A_\nu(x_2)}$$

The state of an electron w/ momentum p_A, spin r_A and a positron w/ momentum p_B, spin r_B is $\sqrt{2\omega_{p_A} 2\omega_{p_B}}\; a_{p_A}^{r_A\dagger} b_{p_B}^{r_B\dagger}|0\rangle$. Note that switching the electron with the positron gives a minus sign. As long as we keep the same order throughout the calculation we'll be fine. Only relative minus signs are important.

The matrix element we need is

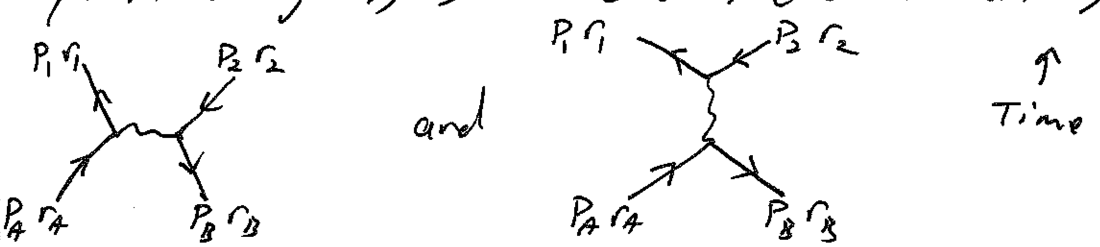
$$\frac{(-ie)^2}{2!}\sqrt{2\omega_{p_A} 2\omega_{p_B} 2\omega_{p_1} 2\omega_{p_2}}\int d^4x_1\, d^4x_2$$

$$\langle 0|a_{p_1}^{r_1} b_{p_2}^{r_2} :\bar{\psi}\gamma^\mu\psi(x_1)\,\bar{\psi}\gamma^\nu\psi(x_2): a_{p_A}^{r_A\dagger} b_{p_B}^{r_B\dagger}|0\rangle\, \overbrace{A_\mu(x_1)A_\nu(x_2)}$$

This time, to annihilate the $b_{p_2}^{r_2}$ we can use the $b_k^{s\dagger}$ in $\psi(x_1)$ or $\psi(x_2)$. To annihilate the $b_{p_B}^{r_B\dagger}$ we can use the b_k^s in $\bar{\psi}(x_1)$ or $\bar{\psi}(x_2)$

Everything proceeds as before, except this time the ingoing electron-positron pair could be annihilated by a $\psi(x)$ and $\bar{\psi}(x)$ respectively at either the same vertex or at different vertices. Depending on that choice, the outgoing electron-positron pair can be created by a $\bar{\psi}(x)$ and $\psi(x)$ respectively at the same or different vertices.

These two possibilities correspond to two Feynman diagrams, as for electron-electron scattering:



In the first diagram, <u>either</u> $\bar{\psi}(x_1)$ annihilates e_B^+
$\psi(x_2)$ annihilates e_A^-
$\psi(x_1)$ creates e_2^+
$\bar{\psi}(x_2)$ creates e_1^-

or $(x_1 \leftrightarrow x_2)$. Again, the $\frac{1}{2!}$ is cancelled by two identical contributions from exchanging $x_1 \leftrightarrow x_2$.

In the second diagram <u>either</u> $\bar{\psi}(x_1)$ annihilates e_B^+
$\psi(x_1)$ annihilates e_A^-
$\bar{\psi}(x_2)$ creates e_1^-
$\psi(x_2)$ creates e_2^+

or $(x_1 \leftrightarrow x_2)$.

This the $\bar{\Psi}\, b_{p_B}^{r_B\,\dagger}|0\rangle$ gives a factor of $\bar{v}^{r_B}(p_B)$

$\Psi\, a_{p_A}^{r_A\,\dagger}|0\rangle$ gives a $u^{r_A}(p_A)$

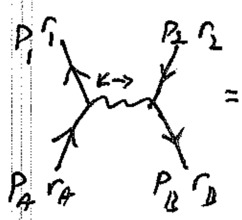
$\langle 0|\, b_{p_2}^{r_2}\, \Psi$ gives a $v^{r_2}(p_2)$

$\langle 0|\, a_{p_1}^{r_1}\, \bar{\Psi}$ gives a $u^{r_1}(p_1)$

Keeping track of the matrix structure of the Dirac spinors and γ-matrices again means starting at the tip of the fermion arrow and working against the arrow.

- An ingoing positron comes with a factor of $\bar{v}^r(p)$
- An outgoing positron comes with a factor of $v^r(p)$.

We can now read off the contributions to the scattering matrix element:

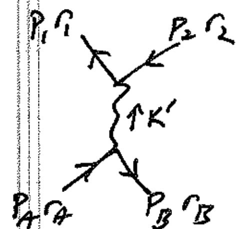


$$= (-ie)^2\, \bar{u}^{r_1}(p_1)\gamma^\mu u^{r_A}(p_A)\ \bar{v}^{r_B}(p_B)\gamma^\nu v^{r_2}(p_2)\, \frac{(-i g_{\mu\nu})}{k^2+i\epsilon}$$
$$\times (2\pi)^4 \delta^4(p_1+p_2-p_A-p_B)$$

where $k = p_A - p_1 = p_2 - p_B$

Note that our conventions are still that ingoing momenta flow into the vertex and outgoing momenta flow out of the vertex, opposite the direction of the positron fermion arrows.

The second diagram is



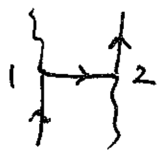
from normal ordering

$$= -(-ie)^2\, \bar{u}^{r_1}(p_1)\,\gamma^\mu\, v^{r_2}(p_2)\; \bar{v}^{r_B}(p_B)\,\gamma^\nu\, u^{r_A}(p_A)\; \frac{(-i g_{\mu\nu})}{k'^2 + i\epsilon}$$
$$\times (2\pi)^4\, \delta^4(p_1 + p_2 - p_A - p_B)$$

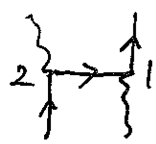
where $k' = p_A + p_B = p_1 + p_2$

The next process we'll consider is <u>Compton Scattering</u>: electron + photon $\rightarrow$ electron + photon.

This time two Wick diagrams contribute:



and



Integrating over d^4x_1 and d^4x_2 these diagrams are equivalent and cancel the $\frac{1}{2!}$ from the Wick expansion.

The sum of the two diagrams is then

$$(-ie)^2 \int d^4x_1\, d^4x_2 \;:\bar{\psi} \not{A} \psi(x_1)\; \bar{\psi} \not{A} \psi(x_2):$$

The electron-photon state is $\sqrt{2\omega_{p_A}\, 2\omega_{p_B}}\; a^{r_A\dagger}_{e,p_A}\, a^{r_B\dagger}_{\gamma,p_B} |0\rangle$.

We have included the subscript e or γ to distinguish the creation operators for the electron and the photon.

We want the matrix element $\sqrt{2\omega_{p_A} 2\omega_{p_B} 2\omega_{p_1} 2\omega_{p_2}}$

$\times (-ie)^2 \int d^4x_1\, d^4x_2 \langle 0 | a^{r_1}_{e,p_1} a^{r_2}_{\gamma,p_2} : \bar{\Psi} \not{A} \Psi(x_1) \bar{\Psi} \not{A} \Psi(x_2) : a^{r_A\dagger}_{e,p_A} a^{r_B\dagger}_{\gamma,p_B} | 0 \rangle$

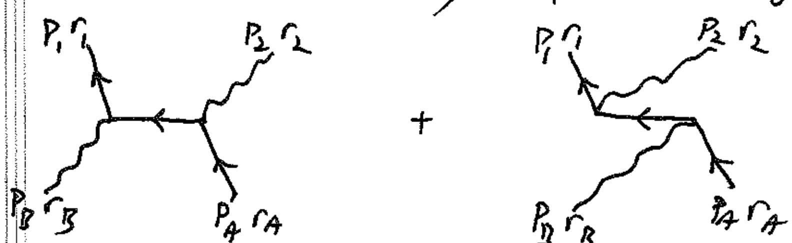
The $a^{s}_{e,k}$ in $\Psi(x_2)$ annihilates the $a^{r_A\dagger}_{e,p_A}$ and gives a $u^{r_A}(p_A)$.

The $a^{s\dagger}_{e,k}$ in $\bar{\Psi}(x_1)$ annihilates the $a^{r_1}_{e,p_1}$ and gives a $\bar{u}^{r_1}(p_1)$

The $a^{s}_{\gamma,k}$ from $A^{\mu}(x_1)$ annihilates the $a^{r_B\dagger}_{\gamma,p_B}$ and gives a $\varepsilon^{\mu(r)}(p_B)$
and $a^{s\dagger}_{\gamma,k}$ from $A^{\mu}(x_2)$ " " $a^{r_2}_{\gamma,p_2}$ and gives a $\varepsilon^{\mu(r)}(p_2)^*$

or
$(x_1 \Leftrightarrow x_2)$

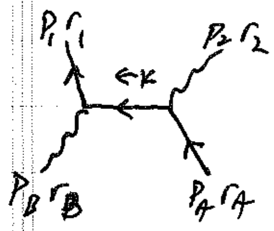
The two corresponding Feynman diagrams are



The internal fermion line $\xleftarrow{\leftarrow k}$ comes from the contraction $\overbrace{\Psi(x_1)\, \bar{\Psi}(x_2)}$ and gives a factor $\frac{i(\not{k}+m)}{k^2-m^2+i\epsilon}$

Note that for the fermion line we are keeping track of the direction of momentum flow, which we are defining <u>along</u> the fermion arrow.

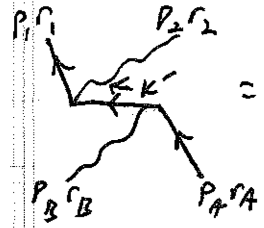
We can now evaluate the Feynman diagrams. Keeping track of the matrix structure of the Dirac spinors and γ-matrices again means starting at the tip of the fermion arrow and working backwards:



$$= (-ie)^2\, \bar{u}^{r_1}(p_1)\,\gamma^\mu\, \frac{i(\not{k}+m)}{k^2-m^2+i\epsilon}\,\gamma^\nu\, u^{r_A}(p_A)\, \varepsilon_\nu^{(r_2)}(p_2)^*\, \varepsilon_\mu^{(r_B)}(p_B)$$

$$\times (2\pi)^4\, \delta^4(p_1+p_2-p_A-p_B)$$

where $k = p_A - p_2 = p_1 - p_B$.



$$= (-ie)^2\, \bar{u}^{r_1}(p_1)\,\gamma^\mu\, \frac{i(\not{k}'+m)}{k'^2-m^2+i\epsilon}\,\gamma^\nu\, u^{r_A}(p_A)\, \varepsilon_\mu^{(r_2)}(p_2)^*\, \varepsilon_\nu^{(r_B)}(p_B)$$

$$\times (2\pi)^4\, \delta^4(p_1+p_2-p_A-p_B)$$

We can summarize these calculations with some basic <u>Feynman Rules for QED</u>:

<u>External Lines</u>

<u>Fermions:</u> Momentum along charge flow!

$\uparrow_{r_A, p_A} = u^{r_A}(p_A)$ ingoing e^- $\qquad$ $\uparrow^{r_1, p_1} = \bar{u}^{r_1}(p_1)$ outgoing e^-

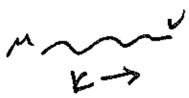
Momentum opposite charge flow

$\downarrow_{r_A, p_A} = \bar{v}^{r_A}(p_A)$ ingoing e^+ $\qquad$ $\downarrow^{r_1, p_1} = v^{r_1}(p_1)$ outgoing e^+

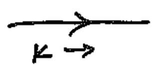
Photons:  $= \varepsilon_\mu^{(r)}(p)$  $= \varepsilon_\mu^{(r)}(p)^*$

μ, r, p ingoing photon μ, r, p outgoing photon

Propagators (internal lines)

Photons: 

$$\frac{-i g_{\mu\nu}}{k^2 + i\epsilon}$$

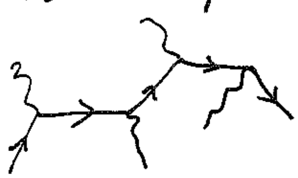
Fermions: 

$$\frac{i(\not{k} + m)}{k^2 - m^2 + i\epsilon}$$

Vertices

$-ie\gamma^\mu$

These rules are enough to calculate <u>any</u> <u>tree level</u> diagram in QED, ie any diagram with no closed loop.



Tree level

One-loop

 ,  Two-loop, etc.