1/11

Scattering Theory

Until now we have studied free quantum field theories. The harmonic oscillator modes were interpreted as particles, and multiparticle states were eigenstates of the Hamiltonian. This implies that there is no scattering in free theories — particles just do their own thing. In the absence of gravity, there would be no way for us to tell whether or not there were free fields around, so the theories we have studied so far are pretty uninteresting from an experimental standpoint.

Terms in the Lagrangian that are more than quadratic in the fields induce interactions.

Examples of interacting QFT's:

Quantum Electrodynamics

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(i\not{\partial} - e\not{A} - m)\Psi$$

Interaction Term

Yukawa's theory of Meson-Nucleon Interactions (simplified)

$$\mathcal{L} = \underbrace{\bar{\Psi}(i\not{\partial} - m)\Psi + \frac{1}{2}(\partial_\mu\phi)^2 - \frac{\mu^2}{2}\phi^2}_{\text{Free field Lagrangian}} + \underbrace{\lambda\bar{\Psi}\phi(a + ib\gamma_5)\Psi}_{\substack{\text{Interactions} \\ (\lambda, a, b \text{ constants})}}$$

Higgs field self-interaction (simplified)

$$\mathcal{L} = |\partial_\mu \phi|^2 - m^2 |\phi|^2 + \underbrace{\frac{\lambda}{4}|\phi|^4}_{\text{Interaction}}$$

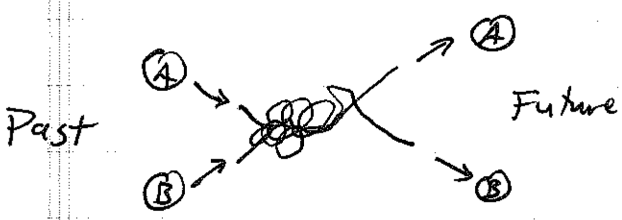
In each of these examples the interactions are responsible for scattering of particles, so we will now turn to scattering theory.

The idea of scattering theory is to take a state with a simple description in the far past and figure out what superposition of simple states in the far future the system evolves into.

Examples from Non-relativistic Quantum Mechanics:

Two particle scattering w/ repulsive interaction

$$H = \frac{p_A^2}{2m_A} + \frac{p_B^2}{2m_B} + V(|\vec{r}_A - \vec{r}_B|) \quad , \; V \geq 0 \, , \; V(\infty) = 0.$$



As far as particles A and B are concerned, in the far past or future a good approximation to the Hamiltonian is:

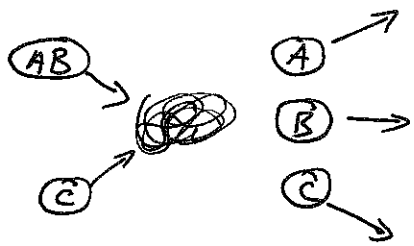
$$H_0 = \frac{p_A^2}{2m_A} + \frac{p_B^2}{2m_B}$$

NRQM w/ Attractive interactions, Bound States:

3 particles A,B,C interact, and A+B can form a bound state AB. The bound state AB can scatter off of C to "ionize" AB into A+B.

Past

$$H_0 = \frac{P_{AB}^2}{2m_{AB}} + \frac{P_c^2}{2m_c}$$



Future

$$H_0 = \frac{P_A^2}{2m_A} + \frac{P_B^2}{2m_B} + \frac{P_c^2}{2m_c}$$

cf. $e^+ + H \to p + e^+ + e^-$

The free Hamiltonian describing the system in the far future looks different than the free Hamiltonian of the far past, but this is only because we have isolated the part of the free Hamiltonian describing the <u>state</u> of the system in the far past/future.

We could have chosen $H_0 = \frac{P_A^2}{2m_A} + \frac{P_B^2}{2m_B} + \frac{P_c^2}{2m_c} + \frac{P_{AB}^2}{2m_{AB}} + \cdots$

describing all possible simple states in the system.

$$H_0 |AB, C\rangle = \left(\frac{P_{AB}^2}{2m_{AB}} + \frac{P_c^2}{2m_c}\right)|AB, C\rangle$$

$$H_0 |A, B, C\rangle = \left(\frac{P_A^2}{2m_A} + \frac{P_B^2}{2m_B} + \frac{P_c^2}{2m_c}\right)|A, B, C\rangle$$

Note that $\langle A, B | AB\rangle = 0$ for consistency of this approach.

<u>Formalism</u>: Let H be the full interacting Hamiltonian and $\mathcal{H}$ the Hilbert space of the interacting system. Suppose in the far past/future we can describe the system by a "free" Hamiltonian H_0 w/ Hilbert space $\mathcal{H}_0$.

Let $|\psi\rangle \in \mathcal{H}_0$ be a simple state in the past. There is a $|\psi\rangle^{in} \in \mathcal{H}$ in the full interacting theory that looks like $|\psi\rangle$ in the past.

Similarly, there is a $|\varphi\rangle \in \mathcal{H}_0$ describing the free system in the future, and a $|\varphi\rangle^{out} \in \mathcal{H}$ in the full interacting system that looks like $|\varphi\rangle$ in the future.

We label states by what they look like in the simple description of the past or future.

We want to know ${}^{out}\langle\varphi|\psi\rangle^{in}$. We can define a <u>scattering matrix</u> S, which is an operator on the simple Hilbert space $\mathcal{H}_0$, such that

$$\boxed{\langle\varphi|S|\psi\rangle \equiv {}^{out}\langle\varphi|\psi\rangle^{in}}$$

Our goals are: 1) A way to obtain the simple Hilbert space $\mathcal{H}_0$ from the full Hamiltonian H.
2) A way to calculate S.

If the interactions are weak then we have a hope of perturbatively approaching these problems. Otherwise they are generally <u>very</u> difficult. Although few people doubt it, the existence of a proton as a bound state in <u>Quantum Chromodynamics</u> has not been proven.

For now we'll stick to weakly interacting systems, so that the interaction terms in the Lagrangian multiply small coupling constants, and we'll develop a perturbative scattering theory.

<u>Time dependent perturbation theory</u>

Suppose $H = H_0 + H'(t)$ in the <u>Schrödinger Picture</u>:

$$i\frac{d}{dt}|\Psi(t)\rangle_s = H(p_s, q_s, t)\,|\Psi(t)\rangle_s \qquad \underline{\text{Sch. Eqn}}$$

p_s, q_s indep. of time in Schrödinger picture.

Time evolution operator: $U(t,t')$, defined s.t.

$$|\Psi(t)\rangle_s = U(t,t')\,|\Psi(t')\rangle_s$$

<u>Sch. Eqn</u> $\Longrightarrow$ $i\frac{d}{dt}U(t,t') = H(p_s, q_s, t)\,U(t,t')$

Initial Condition: $U(t,t')\big|_{t=t'} = 1.$

Properties of $U(t,t')$:

1) ${}_S\langle\Psi(t)|\Psi(t)\rangle_S = 1 \rightarrow U(t,t')^\dagger = U(t,t')^{-1}$

2) Composition $U(t,t')U(t',t'') = U(t,t'')$

3) $U(t,t') = U(t',t)^{-1}$ follows from (2) w/ $t''=t$.

<u>Heisenberg Picture</u>: States constant in time, operators vary w/ time.

(Note that so far our description of field theory has been in the Heisenberg picture.)

$$|\Psi(t)\rangle_H = |\Psi(0)\rangle_H = |\Psi(0)\rangle_S$$

explicit t-dependence

If $A_S(t)$ is a Schrödinger picture operator, define the Heisenberg picture operator $A_H(t)$ as follows:

$$
\begin{aligned}
{}_S\langle\varphi(t)|\, A_S(t)\, |\psi(t)\rangle_S &= {}_H\langle\varphi(t)|\, A_H(t)\, |\Psi(t)\rangle_H \\
&= {}_S\langle\varphi(0)|\, A_H(t)\, |\Psi(0)\rangle_S \\
&= {}_S\langle\varphi(t)|\, U(0,t)^\dagger A_H(t) U(0,t)\, |\Psi(t)\rangle_S
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow A_H(t) &= U(0,t)\, A_S(t)\, U(0,t)^\dagger \\
&= U(t,0)^\dagger A_S(t)\, U(t,0)
\end{aligned}
$$

Functions of operators:

$$A_H(t)^2 = U(t,0)^\dagger A_S(t)\, U(t,0)\, U(t,0)^\dagger A_S(t)\, U(t,0)$$
$$= U(t,0)^\dagger A_S(t)^2\, U(t,0)$$

More generally, $f\big(A_H(t)\big) = U(t,0)^\dagger f\big(A_S(t)\big)\, U(t,0)$

<u>Interaction Picture</u>: $H(p,q,t) = H_0(p,q) + H'(p,q,t)$

Free Hamiltonian ↗ ↖ Interaction

$$|\psi(t)\rangle_I \equiv e^{iH_0(p_s,q_s)t}\,|\psi(t)\rangle_S$$

If $H'=0$ then $|\psi(t)\rangle_I$ is independent of time

If $A_S(t)$ is a Schrödinger picture operator, define the corresponding interaction picture operator $A_I(t)$ by:

$$_S\langle \varphi(t)|\, A_S(t)\, |\psi(t)\rangle_S = {}_I\langle \varphi(t)|\, A_I(t)\, |\psi(t)\rangle_I$$

$$\Rightarrow A_I(t) = e^{iH_0(p_s,q_s)t}\, A_S(t)\, e^{-iH_0(p_s,q_s)t}$$

$$i\frac{d}{dt}|\psi(t)\rangle_I = i\frac{d}{dt}\left(e^{iH_0(p_s,q_s)t}\,|\psi(t)\rangle_S\right)$$

From Sch. eqn

$$= e^{iH_0(p_s,q_s)t}\left(-H_0(p_s,q_s) + H(p_s,q_s,t)\right)|\psi(t)\rangle_S$$

$$= e^{iH_0(p_s,q_s)t}\left(H'(p_s,q_s,t)\right)e^{-iH_0(p_s,q_s)t}\,|\psi(t)\rangle_I$$

$$i \frac{d}{dt} |\psi(t)\rangle_I = H'(p_I, q_I, t) \, |\psi(t)\rangle_I$$

$$\equiv H_I(t) \, |\psi(t)\rangle_I$$

Define $U_I(t,t')$ s.t. $|\psi(t)\rangle_I = U_I(t,t') \, |\psi(t')\rangle_I$

$$i \frac{d}{dt} U_I(t,t') = H_I(t) \, U_I(t,t')$$

Initial condition: $U_I(t,t')\big|_{t=t'} = 1.$

<u>Properties of $U_I(t,t')$:</u>

$$U_I^+(t,t') = U_I(t,t')^{-1}$$
$$U_I(t,t') \, U_I(t',t'') = U_I(t,t'')$$
$$U_I(t,t') = U_I(t',t)^{-1}$$
$$U_I(t,0) = e^{iH_0(p_S, q_S)t} \, U(t,0)$$

<u>Scattering in the Interaction Picture</u>

<u>Far past</u>: $H_I \approx 0 \rightarrow$ states constant, but p's, q's change.

<u>During scattering</u>: $H_I \neq 0 \rightarrow$ states change

<u>Far future</u>: $H_I \approx 0 \rightarrow$ states constant again.

Scattering Matrix:

$$\langle \varphi | S | \psi \rangle \equiv {}^{out}\langle \varphi | \psi \rangle^{in} = {}_I\langle \varphi(0) | \psi(0) \rangle_I$$

$$= {}_I\langle \varphi(\infty) | U_I(\infty, -\infty) | \psi(-\infty) \rangle_I$$

$$= \langle \varphi | U_I(\infty, -\infty) | \psi \rangle$$

$$\longrightarrow \boxed{S = U_I(\infty, -\infty)}$$

The scattering matrix is the same as the interaction picture time evolution operator from $t = -\infty$ to $t = +\infty$.

We need to solve $i \frac{d}{dt} U_I(t,t') = H_I(t)\, U_I(t,t')$

$$U_I(t,t) = 1.$$

If $[H_I(t), H_I(t')] = 0 \quad \forall\, t, t'$ we would have

$$U_I(t,t') = \exp\left[-i \int_{t'}^{t} d\tilde{t}\, H_I(\tilde{t})\right]$$

But in general this is not true. The <u>time ordering</u> operator allows us to cure the problem w/ noncommuting Hamiltonians at unequal times.

Recall $T\left(A_1(t_1) \cdots A_n(t_n)\right)$ reorders $A_1(t_1) \cdots A_n(t_n)$ so that later times are to the left of early times.

10/11

Consider $T\exp\left[-i\int_{t'}^{t} dt''\, H_I(t'')\right]$, $t > t'$.

$$i\frac{d}{dt} T\exp\left[-i\int_{t'}^{t} dt''\, H_I(t'')\right] = H_I(t)\; T\exp\left[-i\int_{t'}^{t} dt''\, H_I(t'')\right]$$

Since t is the latest time appearing in the integral we were able to pull $H_I(t)$ out of the time-ordering symbol.

Also, $T\exp\left[-i\int_{t'}^{t''} H_I(t'')\right] = 1$.

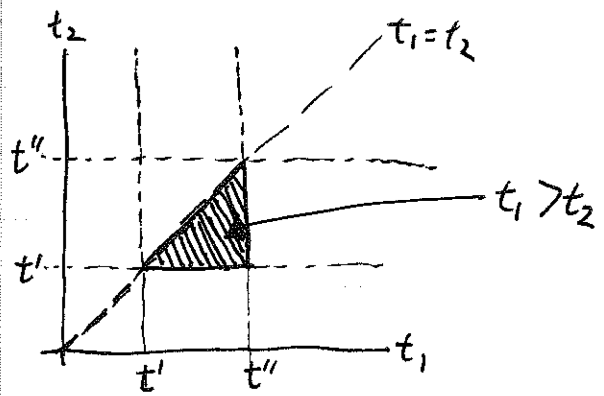
Hence, we identify $\boxed{U_I(t,t') = T\exp\left[-i\int_{t'}^{t} dt''\, H_I(t'')\right]}$

(if $t > t'$.) This is known as Dyson's formula.

For example, consider the second order term in the expansion of the exponential:

$$\frac{(-i)^2}{2!}\int_{t'}^{t} dt_1 \int_{t'}^{t} dt_2\; T\left[H_I(t_1)\, H_I(t_2)\right]$$

$$= (-i)^2 \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2\; H_I(t_1)\, H_I(t_2)$$



integration region for $H_I(t_1)\, H_I(t_2)$.

11/11

Generic term in expansion of the exponential:

$$\frac{(-i)^n}{n!}\int_{t'}^{t} dt_1 \int_{t'}^{t} dt_2 \cdots \int_{t'}^{t} dt_n \; T\left[H_I(t_1)\cdots H_I(t_n)\right]$$

$$= (-i)^n \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 \cdots \int_{t'}^{t_{n-1}} dt_n \; H_I(t_1)\cdots H_I(t_n)$$

Integration is over a wedge in the $(t_1, \ldots, t_n)$ hyperplane.

We now see why time-ordered products of operators are going to be important — they appear in the interaction picture time evolution operator.