Canonical Quantization of the Dirac Spinor Field

So far we have developed a quantum theory of free scalar fields. We have understood why the theory describes multiparticle states, and how causality is preserved.

Let's try to follow the same rules for the Dirac spinor field. Note that $\psi(x)$ is no longer a single particle wavefunction. It is a 4-component field that transforms as a Dirac spinor under Lorentz transformations.

We begin with a Lagrangian that is:

1) Lorentz invariant (scalar)
2) Quadratic in ψ, ψ^+
3) Contains minimal # of derivatives
4) Real (or at least makes the action real)
5) Has no γ^5's (for simplicity.)

We know how to make Lorentz invariant bilinears:

$\bar{\psi}\psi$, $\bar{\psi}\not{\partial}\psi$, $\bar{\psi}\partial_\mu\partial^\mu\psi$, etc — all scalars.

Consider $\int d^4x\, \bar{\psi}\not{\partial}\psi$. Its complex conjugate is

$$\left(\int d^4x\, \bar{\psi}\not{\partial}\psi\right)^* = \int d^4x\, \psi^+ \overleftarrow{\partial_\mu} \gamma^0\gamma^\mu\psi \underset{\text{by parts.}}{=} -\int d^4x\, \bar{\psi}\not{\partial}\psi$$

Hence $\int d^4x\, \bar{\Psi} \not{\partial} \Psi$ is purely imaginary.
But $\int d^4x\, i \bar{\Psi} \not{\partial} \Psi$ is then real.

Also, $\left(\int d^4x\, \bar{\Psi}\Psi\right)^* = \int d^4x\, \bar{\Psi}\Psi$ — real.

So, $\boxed{\mathcal{L} = \pm \left(i \bar{\Psi} \not{\partial} \Psi + b \bar{\Psi}\Psi\right)}$, b real.

<u>Hamiltonian</u>: $\Pi_{\bar{\Psi}} = \dfrac{\partial \mathcal{L}}{\partial(\partial_0 \bar{\Psi})} = 0$

$$\Pi_\Psi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \Psi)} = \pm i \bar{\Psi}\gamma^0 = \pm i \Psi^\dagger$$

$$\mathcal{H} = \partial_0 \bar{\Psi}\, \Pi_{\bar{\Psi}} + \Pi_\Psi\, \partial_0 \Psi - \mathcal{L}$$

$$= \pm i \bar{\Psi}\gamma^0 \partial_0 \Psi \mp \left(i \bar{\Psi}\gamma^0\partial_0\Psi + i\bar{\Psi}\gamma^i\partial_i\Psi + b\bar{\Psi}\Psi\right)$$

$$= \mp i \bar{\Psi}\gamma^i\partial_i\Psi \mp b\bar{\Psi}\Psi$$

$$= \mp \bar{\Psi}\gamma^0 \underbrace{(i\vec{\alpha}\cdot\nabla + b\gamma^0)}\Psi \quad , \text{where } \alpha^i = \gamma^0\gamma^i.$$

Looks like the Dirac Hamiltonian appearing in the Dirac eqn if $b = -m$.

So we'll write $\mathcal{L} = \pm\left(i\bar{\Psi}\not{\partial}\Psi - m\bar{\Psi}\Psi\right)$

where the overall sign will be determined by demanding that the Hamiltonian is bounded below. (which we will see is only possible if the Dirac field describes fermions).

The Euler-Lagrange Eqs: treat ψ and $\bar{\psi}$ as independent. (Or equivalently we can treat ψ and $\psi^\dagger$ as independent.)

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} - \frac{\partial \mathcal{L}}{\partial \psi} = 0 \rightarrow \pm\left(i\bar{\psi}\overleftarrow{\not\partial} + m\bar{\psi}\right) = 0$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} - \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = 0 \rightarrow \mp\,(i\not\partial - m)\psi = 0$$

Self-Consistency: $0 = \left(i\bar{\psi}\overleftarrow{\not\partial} + m\bar{\psi}\right)^\dagger$

$$= -i\gamma^{\mu\dagger}\gamma^0\partial_\mu\psi + m\gamma^0\psi$$

$$= -\gamma^0\left(i\gamma^0\gamma^{\mu\dagger}\gamma^0\partial_\mu\psi - m\psi\right)$$

$$= -\gamma^0\left(i\gamma^\mu\partial_\mu\psi - m\psi\right)$$

Multiplying on the left by $(-\gamma^0)$, we reproduce the $\bar{\psi}$ equation of motion, which we recognize as the Dirac equation.

Following our noses, we <u>canonically quantize</u>:

$$[\psi_a(x), \Pi_b^{\psi}(y)] = i\delta^3(\vec{x}-\vec{y})\delta_{ab} \text{ at equal times } x^0=y^0$$

(a, b are Dirac spinor indices $\in 1,2,3,4$)

$\Pi_b^{\bar{\psi}}(x) = 0$, so we <u>can't</u> also impose $[\bar{\psi}, \Pi^{\bar{\psi}}] = i\delta\ldots$

That's okay, because the Dirac equation is first order in time derivatives. A complete set of initial values is given by $\psi(\vec{x},0)$ and $\Pi^{\psi}(\vec{x},0) \stackrel{\pm}{=} i\psi^{\dagger}$. There is no need to define a conjugate momentum to $\bar{\psi}$.

So, with the Dirac Lagrangian we would impose

$$\boxed{[\psi_a(\vec{x},t), \pm i\psi_b^{\dagger}(\vec{y},t)] = i\delta^3(\vec{x}-\vec{y})\delta_{ab}}$$

The next step is to decompose $\psi(x)$ in plane wave solutions.

We can anticipate the appearance of harmonic oscillator creation and annihilation operators because each component of $\psi(x)$ satisfies the Klein-Gordon equation, like the scalar field.

Recall the positive frequency plane wave sol'ns with momentum in the x^3-direction, in Weyl basis:

$k^0>0$, helicity $+\frac{1}{2}$: $\psi(x) = e^{-ik\cdot x}\begin{pmatrix}\sqrt{E-k^3}\\0\\\sqrt{E+k^3}\\0\end{pmatrix} \equiv e^{-ik\cdot x}\,u_1(k)$

$k^0>0$, helicity $-\frac{1}{2}$: $\psi(x) = e^{-ik\cdot x}\begin{pmatrix}0\\\sqrt{E+k^3}\\0\\\sqrt{E-k^3}\end{pmatrix} = e^{-ik\cdot x}\,u_2(k)$

where $E = k^0 = \omega_k = \sqrt{(k^3)^2+m^2}$

There are also negative frequency plane wave solutions obtained by boosting the negative freq. rest frame solution:

Rest frame: $\psi_{+1/2}(x) = e^{imt}\sqrt{m}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}$, $\psi_{-1/2}(x) = e^{imt}\sqrt{m}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}$

Boosted:

helicity $+1/2$: $\psi(x) = e^{ik\cdot x}\begin{pmatrix}\sqrt{E-k^3}\\0\\-\sqrt{E+k^3}\\0\end{pmatrix} \equiv e^{ik\cdot x}\,v_1(k)$

helicity $-1/2$: $\psi(x) = e^{ik\cdot x}\begin{pmatrix}0\\\sqrt{E+k^3}\\0\\-\sqrt{E-k^3}\end{pmatrix} \equiv e^{ik\cdot x}\,v_2(k)$

where $E = k^0 = \omega_k = \sqrt{(k^3)^2+m^2}$,

Note that as written $k^0>0$, but we call these the negative freq. sol'ns.

Completeness Relations and Orthogonality

Consider the rest frame sol'ns (in Weyl basis):

$$u_1(k) = \sqrt{m}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \quad u_2(k) = \sqrt{m}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \quad k=(m,\vec{0})$$

$$\boxed{\bar{u}^r(k)u^s(k) = u^{r\dagger}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}u^s = 2m\,\delta^{rs}}$$

Orthogonality

Since $\bar{u}^r u^s$ is a Lorentz scalar, this is also valid in a boosted frame.

$$\sum_{r=1,2} u^r(k)\bar{u}^r(k) = m\begin{pmatrix}1\\0\\1\\0\end{pmatrix}(1\,0\,1\,0) + m\begin{pmatrix}0\\1\\0\\1\end{pmatrix}(0\,1\,0\,1)$$

$$= m\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} = m(\gamma^0 + 1)$$

$$= k^0\gamma^0 + m\,1$$

We can guess the form of this completeness relation in a boosted frame:

$$\boxed{\sum_{r=1}^{2} u^r(k)\bar{u}^r(k) = \not{k} + m}$$

Completeness

Check: $\left(\sum_r u^r(k)\bar{u}^r(k)\right)u^s(k) = \sum_r u^r(k)\cdot 2m\,\delta^{rs} = 2m\,u^s(k)$ ✓

We can also check that $\boxed{\bar{u}^r(k)\,v^s(k) = 0}$, so

$$\left(\sum_{r=1}^{2} u^r(k)\bar{u}^r(k)\right)v^s(k) = 0 \quad ✓$$

Similarly, $\boxed{\bar{v}^r(k)\, v^s(k) = -2m\,\delta^{rs}}$

$\boxed{\bar{v}^r(k)\, u^s(k) = 0}$

Orthogonality

$$\boxed{\sum_{r=1}^{2} v^r(k)\,\bar{v}^r(k) = \not{k} - m}$$

Completeness

Now we're ready to write down the decomposition of $\psi(x)$ in plane waves. We have Fourier coefficients $a^r_{\vec{k}}$, $b^r_{\vec{k}}$ for each sol'n $(r=1,2)$.

$$\boxed{\psi(x) = \int \frac{d^3 k}{(2\pi)^3\sqrt{2\omega_k}} \left(e^{-ik\cdot x} \sum_r a^r_{\vec{k}}\, u^r(k) + e^{ik\cdot x} \sum_r b^r_{\vec{k}}\, v^r(k) \right)}$$

The factor of $\frac{1}{(2\pi)^3\sqrt{2\omega_k}}$ is conventional, and could have been absorbed in $a^r_{\vec{k}}$ and $b^r_{\vec{k}}$. It is convenient for the same reasons as the scalar field.

Note that we have called the coefficient of $e^{ik\cdot x}$ $b^r_{\vec{k}}$ and not $b^{r\dagger}_{\vec{k}}$. This is because we will see next that both $a^r_{\vec{k}}$ and $b^r_{\vec{k}}$ are annihilation operators. This will be the downfall of the equal time commutation relation approach for fermions, as we will see next.