Physics 721, Fall 2010
Problem Set 2
Due Monday, September 20.

1. Dirac Bilinears

If $\Lambda^\mu_\nu$ describes a Lorentz transformation, such that $x^\mu \rightarrow \Lambda^\mu_\nu x^\nu$, then how do the following transform under the Lorentz transformation? You should derive your results by explicit computation.

a) $\partial_\mu \left( \bar{\psi}(x) \gamma^\mu \psi(x) \right)$, where $\psi(x)$ is a Dirac spinor field?

b) $\bar{\psi}(x) [\gamma^\mu, \gamma^\nu] \psi(x)$?

2. Chirality

Any Dirac spinor can be decomposed into a left-handed and a right-handed part by using the chirality projection operators,

$$ P_L = \frac{1 - \gamma^5}{2}, \quad P_R = \frac{1 + \gamma^5}{2}. $$

Using the properties of $\gamma^5$ show that:

a) $P_L^2 = P_L, \quad P_R^2 = P_R, \quad P_LP_R = P_RP_L = 0$.

b) Given a Dirac spinor $\psi$ define its left-handed part as $\psi_L \equiv P_L \psi$ and its right-handed part as $\psi_R \equiv P_R \psi$. Show that under a Lorentz transformation of $\psi$, the left and right-handed components of $\psi$ transform independently. This implies that the Dirac spinor forms a reducible representation of the Lorentz group.

c) By acting on the Dirac equation $(i\partial - m) \psi = 0$ with $P_L$ and with $P_R$, rewrite the Dirac equation in terms of a coupled set of equations for $\psi_L$ and $\psi_R$.

d) Show that the equations for $\psi_L$ and $\psi_R$ decouple when $m \rightarrow 0$. 

3. Solution to free Dirac equation with momentum in arbitrary direction

The solution to the Dirac equation for an electron moving in the $x^3$-direction with momentum $p^3$ is, in the Weyl basis,

$$\psi(x) = u(p) e^{ip^\mu x_\mu},$$

where,

$$u(p) = \begin{pmatrix} \sqrt{E + p^3} \left(\frac{1 - \sigma^3}{2}\right) + \sqrt{E - p^3} \left(\frac{1 + \sigma^3}{2}\right) \xi \\ \sqrt{E + p^3} \left(\frac{1 + \sigma^3}{2}\right) + \sqrt{E - p^3} \left(\frac{1 - \sigma^3}{2}\right) \xi \end{pmatrix},$$

and $\xi$ is a 2-component Pauli spinor.

Show that the above solution can be written as,

$$u(p) = \begin{pmatrix} \sqrt{p^\mu} \sigma_\mu \xi \\ \sqrt{p^\mu} \sigma_\mu \xi \end{pmatrix},$$

where $\sigma^\mu = (1, \vec{\sigma})$, $\sigma^\mu = (1, -\vec{\sigma})$. In this form the solution is valid when the momentum is in an arbitrary direction.

4. $SO(3,1)$ Algebra (Optional Problem)

The generators of the rotation group in three dimensions, $SO(3)$, satisfy the algebra $[T^a, T^b] = i \sum e^{abc} T^c$. The matrices $T^a = \sigma^a / 2, a = 1, 2, 3$, form a representation of the algebra.

The analogous relations for the six generators of Lorentz transformations $J^{\mu\nu}, \mu, \nu = 0, 1, 2, 3$, with $J^{\mu\nu} = -J^{\nu\mu}$, are

$$[J^{\mu\nu}, J^{\rho\sigma}] = i (\eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\sigma} J^{\mu\rho} + \eta^{\mu\sigma} J^{\nu\rho}).$$

These commutation relations define the algebra $SO(3,1)$. Using the properties of the Dirac $\gamma$-matrices, show that the generators of Lorentz transformations in the Dirac spinor representation,

$$S^{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu],$$

satisfy the commutation relations describing the Lorentz algebra.