Complex Variables Cheat Sheet(s)

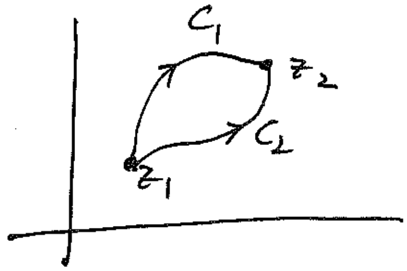
$f(z)$ is an <u>analytic</u> function of the complex variable z <u>in a region R</u> of the complex plane if $f(z)$ is differentiable and single-valued everywhere in R.

$f(z)$ is <u>analytic at the pt z</u> if z is an interior point of some region where $f(z)$ is analytic.

Differentiability implies the Cauchy-Riemann Eqs:
If $f(z) = u(x,y) + i\, v(x,y)$ with $z = x+iy$ and u, v real fns, then

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}\right\} \quad \underline{\text{Cauchy-Riemann Eqs.}}$$

If $f(z)$ is analytic in a simply connected region including a path C in the complex plane, then it follows from the Cauchy-Riemann Eqs that $f(z)\,dz$ is an <u>exact differential</u>, and integrals bet. pts z_1 and z_2 are independent of the path C between them within the region of analyticity of f.

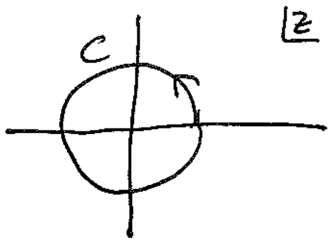


$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

if $f(z)$ is analytic in the interior of $C_1 \cup C_2$.

It also follows that if $f(z)$ is analytic in the interior of a closed loop C then $\oint_C f(z)\,dz = 0$.

Assume $f(z) = z^n$, n = integer.
Let C = counterclockwise loop along unit circle,



Let $z = e^{i\theta}$

$dz = i e^{i\theta}\,d\theta$

So $$\oint_C z^n\,dz = \int_0^{2\pi} e^{in\theta} \cdot i e^{i\theta}\,d\theta$$

$$= \int_0^{2\pi} e^{i(n+1)\theta}\,d\theta$$

$$= i\int_0^{2\pi} \left[\cos(n+1)\theta + i\sin(n+1)\theta\right] d\theta$$

$$= \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = -1 \end{cases}$$

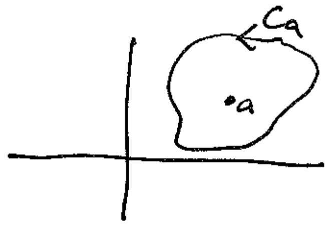
The contour C can be deformed to any contour encircling the origin once, and the integral is as above.

If $f(z)$ has a Laurent expansion about a pt. $z=a$, then

$$\int_{C_a} f(z)\,dz = \int_{C_a} \sum_{n=-\infty}^{\infty} A_n (z-a)^n \, dz$$

$$= 2\pi i \, A_{-1}$$

if C_a is a contour encircling a once.

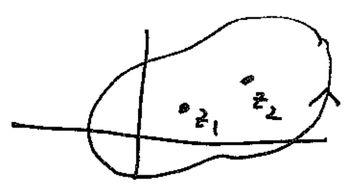


If $f(z)$ is analytic inside a closed contour C except for a finite # of poles, then

$$\boxed{\int_C f(z)\,dz = 2\pi i \sum \text{Residues at poles inside } C}$$

where the residue at a pt a is the coefficient of $(z-a)^{-1}$ in the Laurent expansion, called A_{-1} above.

Example: $\int_C \left[\frac{A}{z-z_1} + \frac{B}{z-z_2}\right] dz$, where C is the contour



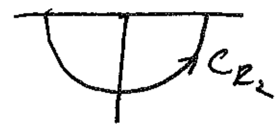
then $\int_C \left[\frac{A}{z-z_1} + \frac{B}{z-z_2}\right] dz = 2\pi i \,(A+B)$.

Integrals involving exponentials over large circular arcs

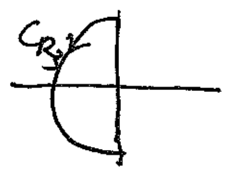
If $f(z) \to 0$ uniformly as $z \to \infty$, $m > 0$, then:

$$\lim_{R_1 \to \infty} \int_{C_{R_1}} e^{imz} f(z)\, dz = 0$$

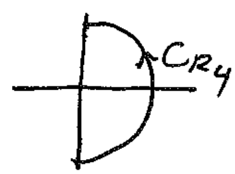
$$\lim_{R_2 \to \infty} \int_{C_{R_2}} e^{-imz} f(z)\, dz = 0$$



$$\lim_{R_3 \to \infty} \int_{C_{R_3}} e^{mz} f(z)\, dz = 0$$

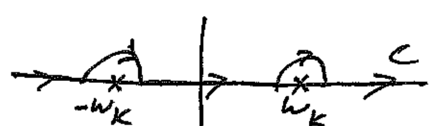


$$\lim_{R_4 \to \infty} \int_{C_{R_4}} e^{-mz} f(z)\, dz = 0$$



We can use this to compute integrals along the real or imaginary axis:

<u>Example</u>: Let $t > 0$, Contour C



$$\int_c dK \, \frac{i e^{-iKt}}{K^2 - w_K^2} = \int_{C - C_{R_2}} dK \, \frac{i e^{-iKt}}{K^2 - w_K^2}$$

$$= -2\pi i \cdot i \left(\frac{e^{-iw_K t}}{2 w_K} - \frac{e^{i w_K t}}{2 w_K} \right)$$

where we used the contour C_{R_2} from above, and the Res. thm.