Canonical Quantization

In quantum mechanics the coordinates and momenta become operators which act on a Hilbert space. The coords and momenta satisfy the commutation relations

$$[q_a, p_b] = i\,\delta_{ab} \qquad (\text{remember } \hbar = 1)$$
$$[q_a, q_j] = [p_a, p_j] = 0$$

Recall our dictionary between classical mechanics and classical field theory:

$$q_a(t) \to \phi_a(x)$$
$$p_a \to \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_a)} \equiv \Pi_a(x)$$

We postulate the <u>equal-time commutation relations</u>

$$\boxed{\begin{aligned} &[\phi_a(t,\vec{x}), \Pi_b(t,\vec{y})] = i\,\delta_{ab}\,\delta^3(\vec{x}-\vec{y}) \\ &[\phi_a(t,\vec{x}), \phi_b(t,\vec{y})] = [\Pi_a(t,\vec{x}), \Pi_b(t,\vec{y})] = 0 \end{aligned}}$$

<u>Exercise</u>: By dimensional analysis put the $\hbar$'s back.

The equal-time commutation relations are also called <u>canonical commutation relations</u>.

Example: Single real scalar field

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2$$

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0 \qquad \text{Euler-Lagrange eqn.}$$

Solutions can be decomposed in plane waves:

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 \sqrt{2\omega_{\vec{k}}}} \left[a_{\vec{k}}\, e^{-i(\omega_{\vec{k}} t - \vec{k}\cdot\vec{x})} + a_{\vec{k}}^{\dagger}\, e^{i(\omega_{\vec{k}} t - \vec{k}\cdot\vec{x})} \right]$$

$$\text{where} \quad \omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}.$$

The factor of $1/\sqrt{2\omega_{\vec{k}}}$ is an arbitrary normalization factor, but will be useful later (because of Lorentz invariance).

$\phi(x)$ is an operator-valued function of spacetime, so we assume the Fourier coefficients $a_{\vec{k}}, a_{\vec{k}}^{\dagger}$ are operators.

The canonical momentum is

$$\Pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi$$

$$= \int \frac{d^3k}{(2\pi)^3}\, i \sqrt{\frac{\omega_{\vec{k}}}{2}} \left[-a_{\vec{k}}\, e^{-ik\cdot x} + a_{\vec{k}}^{\dagger}\, e^{ik\cdot x} \right]$$

where $k\cdot x = \omega_{\vec{k}} t - \vec{k}\cdot\vec{x}$

We can solve for $a_{\vec{k}}$ and $a_{\vec{k}}^{+}$ in terms of $\phi(x)$ and $\Pi(x)$.

$$\boxed{\int d^3x \sqrt{2\omega_{k'}}\, e^{ik'\cdot x}\, \phi(x)}$$

$$= \int d^3x \sqrt{2\omega_{k'}}\, e^{ik'\cdot x} \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} \left[a_k e^{-ik\cdot x} + a_k^+ e^{ik\cdot x}\right]$$

$$= \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{\omega_{k'}}{\omega_k}} \Big[(2\pi)^3 \delta^3(\vec{k}-\vec{k}')\, a_k\, e^{i(\omega_{k'}-\omega_k)t}$$
$$+ (2\pi)^3 \delta^3(\vec{k}+\vec{k}')\, a_k^+\, e^{i(\omega_{k'}+\omega_k)t}\Big]$$

$$\boxed{= a_{\vec{k}'} + a_{-\vec{k}'}^{+}\, e^{2i\omega_{\vec{k}'}t}}$$

using $\int d^3x\, e^{-i(\vec{k}'-\vec{k})\cdot\vec{x}} = (2\pi)^3 \delta^3(\vec{k}-\vec{k}')$

and $\omega_{-\vec{k}} = \omega_{\vec{k}}$

Similarly,

$$\boxed{\int d^3x\, i\sqrt{\frac{2}{\omega_{k'}}}\, e^{ik'\cdot x}\, \Pi(x) = a_{\vec{k}'} - a_{-\vec{k}'}^{+}\, e^{2i\omega_{\vec{k}'}t}}$$

Solving for $a_{\vec{k}'}$, we get

$$\boxed{\begin{aligned} a_{\vec{k}} &= \frac{1}{2}\int d^3x\, e^{i(\omega_{\vec{k}}t-\vec{k}\cdot\vec{x})}\left[\sqrt{2\omega_{\vec{k}}}\,\phi(x) + i\sqrt{\frac{2}{\omega_{\vec{k}}}}\,\Pi(x)\right] \\ a_{\vec{k}}^{+} &= \frac{1}{2}\int d^3x\, e^{-i(\omega_{\vec{k}}t-\vec{k}\cdot\vec{x})}\left[\sqrt{2\omega_{\vec{k}}}\,\phi(x) - i\sqrt{\frac{2}{\omega_{\vec{k}}}}\,\Pi(x)\right] \end{aligned}}$$

We now find the commutation relations for $a_{\vec{k}}, a_{\vec{k}}^{\dagger}$.

$k^0 = \omega_{\vec{k}} \equiv \omega_k$

$k'^0 = \omega_{\vec{k}'} \equiv \omega_{k'}$

$$\boxed{[a_{\vec{k}}, a_{\vec{k}'}]} = \frac{1}{4}\int d^3x\, d^3x'\, e^{ik\cdot \overset{(t,\vec{x})}{x}} e^{ik'\cdot \overset{(t,\vec{x}')}{x'}} \left(i\sqrt{\frac{\omega_k}{\omega_{k'}}}\,[\phi(x), \pi(x')] + i\sqrt{\frac{\omega_{k'}}{\omega_k}}\,[\pi(x), \phi(x')] \right)$$

$$= \frac{1}{4}\int d^3x\, d^3x'\, e^{i(k\cdot x + k'\cdot x')}\cdot i\left(\sqrt{\frac{\omega_k}{\omega_{k'}}}\, i\,\delta^3(\vec{x}-\vec{x}') + \sqrt{\frac{\omega_{k'}}{\omega_k}}\,(-i)\,\delta^3(\vec{x}-\vec{x}')\right)$$

$$= \frac{1}{4}\int d^3x\; e^{i(\omega_k+\omega_{k'})t}\, e^{i(\vec{k}+\vec{k}')\cdot\vec{x}}\left(-\sqrt{\frac{\omega_k}{\omega_{k'}}} + \sqrt{\frac{\omega_{k'}}{\omega_k}}\right)$$

$$= \frac{1}{4}(2\pi)^3\,\delta^3(\vec{k}+\vec{k}')\,e^{2i\omega_k t}\,(-1+1)$$

$$\boxed{=0}$$

Similarly, $\boxed{[a_{\vec{k}}^{\dagger}, a_{\vec{k}'}^{\dagger}] = 0}$ (Exercise)

$$\boxed{[a_{\vec{k}}, a_{\vec{k}'}^{\dagger}]} = \frac{1}{4}\int d^3x\, d^3x'\, e^{ik\cdot x} e^{-ik'\cdot x'}\left(2\sqrt{\frac{\omega_k}{\omega_{k'}}}\,[\phi(x), \pi(x')](-i) + 2i\sqrt{\frac{\omega_{k'}}{\omega_k}}\,[\pi(x), \phi(x')]\right)$$

$$= \frac{1}{4}\int d^3x\, d^3x'\, e^{i(k\cdot x - k'\cdot x')}\cdot 2\left(\sqrt{\frac{\omega_k}{\omega_{k'}}} + \sqrt{\frac{\omega_{k'}}{\omega_k}}\right)\delta^3(\vec{x}-\vec{x}')$$

$$= \frac{1}{2}\int d^3x\, e^{i(\omega_k-\omega_{k'})t}\, e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}}\left(\sqrt{\frac{\omega_k}{\omega_{k'}}} + \sqrt{\frac{\omega_{k'}}{\omega_k}}\right)$$

$$\boxed{= (2\pi)^3\,\delta^3(\vec{k}-\vec{k}')}$$

In summary,

$$\boxed{\begin{aligned} [a_{\vec{k}}, a_{\vec{k}'}] &= [a_{\vec{k}}^\dagger, a_{\vec{k}'}^\dagger] = 0 \\ [a_{\vec{k}}, a_{\vec{k}'}^\dagger] &= (2\pi)^3 \delta^3(\vec{k}-\vec{k}') \end{aligned}}$$

These commutators remind us of the raising and lowering operator commutation relations of the simple harmonic oscillator, but here we have a raising and lowering operator for each $\vec{k}$.

<u>Aside</u>: Lorentz invariant measure

$$\boxed{\int_{k^0>0} \frac{d^4k}{(2\pi)^3} \delta(k^2-m^2)} \quad \text{— manifestly Lorentz invariant}$$

$$= \int_{k^0>0} \frac{d^4k}{(2\pi)^3} \delta\left((k^0)^2 - (\vec{k}^2+m^2)\right)$$

$$= \int_{(k^0>0)} \frac{d^4k}{(2\pi)^3} \frac{1}{2\sqrt{\vec{k}^2+m^2}} \delta\left(k^0 - \sqrt{\vec{k}^2+m^2}\right)$$

using $\displaystyle \int dx\, \delta\left(f(x)\right) = \sum_{\substack{\text{zeros} \\ x_n \text{ of } f(x)}} \frac{\delta(x-x_n)}{|f'(x_n)|}$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\sqrt{\vec{k}^2+m^2}} \quad \boxed{= \int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}}}$$

Integrals against the Lorentz-invariant measure $\frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}}$ will have Lorentz transformation properties dictated solely by the integrand.

Now consider the Hamiltonian,

$$H = \int d^3x \ \frac{1}{2}\left[\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2\right]$$

Since $\phi(x)$ is an operator, so is H.

$$H = \int d^3x \int \frac{d^3k \ d^3k'}{(2\pi)^6\sqrt{2\omega_k 2\omega_{k'}}} \frac{1}{2}\Big\{ e^{-ik\cdot x - ik'\cdot x} a_k a_{k'} \left(-\omega_k\omega_{k'} - \vec{k}\cdot\vec{k}' + m^2\right)$$

$$+ e^{-ik\cdot x + ik'\cdot x} a_k a^\dagger_{k'} \left(\omega_k\omega_{k'} + \vec{k}\cdot\vec{k}' + m^2\right)$$

$$+ e^{ik\cdot x - ik'\cdot x} a^\dagger_k a_{k'} \left(\omega_k\omega_{k'} + \vec{k}\cdot\vec{k}' + m^2\right)$$

$$+ e^{ik\cdot x + ik'\cdot x} a^\dagger_k a^\dagger_{k'} \left(-\omega_k\omega_{k'} - \vec{k}\cdot\vec{k}' + m^2\right)\Big\}$$

where $k^0 = \omega_{\vec{k}}$, $k^{0\prime} = \omega_{\vec{k}'}$. Integrating over $\vec{x}$,

$$H = \frac{1}{2}\int \frac{d^3k \ d^3k'}{(2\pi)^3\sqrt{2\omega_k 2\omega_{k'}}} \Big\{ a_k a_{k'} \, \delta^3(\vec{k}+\vec{k}') \, e^{-i(\omega_k+\omega_{k'})t} \left(-\omega_k\omega_{k'} - \vec{k}\cdot\vec{k}' + m^2\right)$$

$$+ a_k a^\dagger_{k'} \, \delta^3(\vec{k}-\vec{k}') \, e^{-i(\omega_k-\omega_{k'})t} \left(\omega_k\omega_{k'} + \vec{k}\cdot\vec{k}' + m^2\right)$$

$$+ a^\dagger_k a_{k'} \, \delta^3(\vec{k}-\vec{k}') \, e^{i(\omega_k-\omega_{k'})t} \left(\omega_k\omega_{k'} + \vec{k}\cdot\vec{k}' + m^2\right)$$

$$+ a^\dagger_k a^\dagger_{k'} \, \delta^3(\vec{k}+\vec{k}') \, e^{i(\omega_k+\omega_{k'})t} \left(-\omega_k\omega_{k'} - \vec{k}\cdot\vec{k}' + m^2\right)\Big\}$$

$$H = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega_k}\Big\{ a_{\vec k} a_{-\vec k} e^{-2i\omega_k t}\cancel{\left(-\omega_{\vec k}^2 + \vec k^2 + m^2\right)}$$
$$+ a_{\vec k} a_{\vec k}^\dagger \left(\omega_{\vec k}^2 + \vec k^2 + m^2\right)$$
$$+ a_{\vec k}^\dagger a_{\vec k} \left(\omega_{\vec k}^2 + \vec k^2 + m^2\right)$$
$$+ a_{\vec k}^\dagger a_{-\vec k}^\dagger e^{2i\omega_k t}\cancel{\left(-\omega_{\vec k}^2 + \vec k^2 + m^2\right)}\Big\}$$

$$\boxed{H = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\, \omega_{\vec k}\left(a_{\vec k} a_{\vec k}^\dagger + a_{\vec k}^\dagger a_{\vec k}\right)}$$

This looks just like the harmonic oscillator Hamiltonian for each $\vec k$. There is also a contribution to the zero-point energy for each $\vec k$, which leads to an ill-defined constant contribution to H:

Using $[a_{\vec k}, a_{\vec k'}^\dagger] = (2\pi)^3 \delta^3(\vec k - \vec k')$,

$$H = \int \frac{d^3k}{(2\pi)^3}\left(\omega_{\vec k}\, a_{\vec k}^\dagger a_{\vec k}\right) + \underbrace{\frac{1}{2}\int d^3k\, \omega_{\vec k}\, \delta^3(\vec 0)}_{\text{zero-point energy}}$$

Yikes! What is $\delta^3(\vec 0)$? Not to worry:

V = box volume

1. If the system were in a finite-sized box then $(2\pi)^3\delta^3(\vec k - \vec k')$ would be replaced by $V\delta_{\vec k \vec k'}$ over the discrete spectrum of momenta.
2. You can't measure the zero-point energy, anyway (except <u>maybe</u> by gravity — that's the cosmological constant problem.)