

Dirac Field Bilinears

If $\Psi(x)$ is a Dirac field and Γ is a 4×4 matrix, then $\bar{\Psi} \Gamma \Psi$ can be decomposed into a sum of terms which transform covariantly under Lorentz transformations.

The matrices $1, \gamma^\mu, \gamma^{\mu\nu} \equiv \frac{1}{2} [\gamma^\mu, \gamma^\nu] = -i \sigma^{\mu\nu}$,
 $\gamma^{\mu\nu\rho} \equiv \gamma^{\mu\nu} \gamma^\rho, \gamma^{\mu\nu\rho\sigma} \equiv \gamma^{\mu\nu} \gamma^{\rho\sigma}$

form a complete basis of 4×4 matrices. The Lorentz transformation of $\bar{\Psi} \Gamma \Psi$ can be read directly from the index structure of Γ .

$\gamma^{\mu\nu\rho}$ and $\gamma^{\mu\nu\rho\sigma}$ can be simplified by defining $\gamma^5 \equiv i \gamma^0 \gamma^1 \gamma^2 \gamma^3$.

Exercise: Show $\gamma^{\mu\nu\rho\sigma} = -i \epsilon^{\mu\nu\rho\sigma} \gamma^5$
and $\gamma^{\mu\nu\rho} = -i \epsilon^{\mu\nu\rho\sigma} \gamma_\sigma \gamma^5$

Note that under Lorentz transtns. connected to the identity, $\bar{\Psi} \gamma^{\mu\nu\rho\sigma} \Psi$ is a scalar.

Properties of γ^5 :

$$(\gamma^5)^t = -i (\gamma^3)^t (\gamma^2)^t (\gamma^1)^t (\gamma^0)^t$$

Recall $\gamma^0 = \beta = (\gamma^0)^t$

$$\gamma^i = \beta \alpha^i \rightarrow (\gamma^i)^t = (\alpha^i)^t \beta^t = \alpha^i \beta = -\gamma^i$$

$$\text{So, } (\gamma^5)^t = -i (-\gamma^3) (-\gamma^2) (-\gamma^1) \gamma^0 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \gamma^5$$

→ $\boxed{(\gamma^5)^t = \gamma^5}$

$$\begin{aligned}
 (\gamma^5)^2 &= -\gamma_0\gamma_1\gamma_2\gamma_3 \gamma_0\gamma_1\gamma_2\gamma_3 \\
 &= -\gamma_0\gamma_0\gamma_1\gamma_1\gamma_2\gamma_2\gamma_3\gamma_3 \\
 &= +1
 \end{aligned}$$

$$\begin{aligned}
 \{\gamma^5, \gamma^\mu\} &= \gamma^5\gamma^\mu + \gamma^\mu\gamma^5 \\
 &= i\gamma_0\gamma_1\gamma_2\gamma_3\gamma^\mu + i\gamma^\mu\gamma_0\gamma_1\gamma_2\gamma_3 \\
 &= 0
 \end{aligned}$$

(γ^μ is anticommutated through three of $\gamma_0, \gamma_1, \gamma_2, \gamma_3$.)

$$\begin{aligned}
 [\gamma^5, \sigma^{\mu\nu}] &= \frac{i}{2} (\gamma^5 [\gamma^\mu, \gamma^\nu] - [\gamma^\mu, \gamma^\nu] \gamma^5) \\
 &= 0
 \end{aligned}$$

(anticommuting γ^5 through γ^μ and γ^ν)

γ^5 and Chirality

Because $\frac{i}{4}\sigma^{\mu\nu}$ generates Lorentz transformations and $[\gamma^5, \sigma^{\mu\nu}] = 0$, Dirac spinors Ψ with different γ^5 eigenvalues transform without mixing.

The 4-component Dirac spinor representation decomposes into 2-component reps with $\gamma^5 = \pm 1$.

In the Weyl basis, $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$.

$$\sigma^{ij} = \frac{i}{2} \begin{pmatrix} -[\sigma^i, \sigma^j] & \\ & -[\sigma^0, \sigma^j] \end{pmatrix} = \sum_k \epsilon^{ijk} \begin{pmatrix} \sigma^k & \\ & \sigma^k \end{pmatrix}$$

(convention: $\epsilon^{123} = 1$)

Infinitesimal rotation parametrized by

$$\omega^{ij} = \sum_K \epsilon^{ijk} \theta^K$$

$$-\frac{i}{4} \omega^{ij} \sigma_{ij} = -\frac{i}{4} \sum_{ijkl} \epsilon^{ijk} \theta^K \epsilon^{ijl} \begin{pmatrix} \sigma^l & \\ & \sigma^l \end{pmatrix}$$

$$= -\frac{i}{2} \begin{pmatrix} \vec{\theta} \cdot \vec{\sigma} & \\ & \vec{\theta} \cdot \vec{\sigma} \end{pmatrix}$$

using $\sum_{ij} \epsilon^{ijl} \epsilon^{ijk} = 2 \delta^{kl}$

$$\overline{\sigma_{0i}} = \frac{i}{2} [\gamma_0, \sigma_i] = -\frac{i}{2} \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \right]$$

$$= i \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$$

Infinitesimal boost parametrized by $\omega^{0i} = -\frac{v^i}{c}$

$$-\frac{i}{4} (\omega^{0i} \sigma_{0i} + \omega^{i0} \sigma_{i0}) = -\frac{i}{2} \omega^{0i} \sigma_{0i}$$

$$= -\frac{i}{2} \left(-\frac{v^i}{c} \right) \begin{pmatrix} i \sigma^i & 0 \\ 0 & -i \sigma^i \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} -\frac{\vec{v}}{c} \cdot \vec{\sigma} & 0 \\ 0 & \frac{\vec{v}}{c} \cdot \vec{\sigma} \end{pmatrix}$$

$$\exp \left[-\frac{i}{4} \sigma_{\mu\nu} \omega^{\mu\nu} \right] = \exp \left[\frac{\vec{\sigma}}{2} \cdot \begin{pmatrix} -i\vec{\theta} - \frac{\vec{v}}{c} & 0 \\ 0 & -i\vec{\theta} + \frac{\vec{v}}{c} \end{pmatrix} \right]$$

In the Weyl basis, write $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$

where ψ_L, ψ_R each have two components.

Under a Lorentz transformation,

$$\psi \rightarrow \exp\left[-\frac{i}{4} \sigma_{\mu\nu} \omega^{\mu\nu}\right] \psi, \text{ i.e.}$$

$$\psi_L \rightarrow \exp\left[-\frac{i}{2} \vec{\sigma} \cdot \vec{\theta} - \frac{\vec{v} \cdot \vec{\sigma}}{2c}\right] \psi_L$$

$$\psi_R \rightarrow \exp\left[-\frac{i}{2} \vec{\sigma} \cdot \vec{\theta} + \frac{\vec{v} \cdot \vec{\sigma}}{2c}\right] \psi_R$$

ψ_L and ψ_R transform in different irreducible representations of the Lorentz group.

In the Weyl basis, $\gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$

$\begin{pmatrix} \psi_L \\ 0 \end{pmatrix}$ has γ^5 eigenvalue -1 . — called left-handed.

$\begin{pmatrix} 0 \\ \psi_R \end{pmatrix}$ has γ^5 eigenvalue $+1$. — called right-handed.

Free Particle Solutions to the Dirac Equation

We will construct plane-wave solutions in the Weyl basis.

$$\psi(x) = u(p) e^{-i p \cdot x}, \quad p^\mu p_\mu = m^2$$

Positive frequency solutions: $p^0 > 0$.

$$\text{Dirac equation: } (\gamma^\mu p_\mu - m) u(p) = 0$$

In the rest frame of the particle, $p^0 = m$, $p^i = 0$, $i=1,2,3$.

Then the Dirac equation becomes $m(\gamma^0 - 1)u(p) = 0$.

$$m \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} u(p) = 0$$

$$\text{Solutions: } u(p) = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

ξ = arbitrary 2-component constant spinor.

Convenient normalization: $\xi^\dagger \xi = 1$.

To find an arbitrary free particle solution, boost the rest frame solution in the direction of motion.

Example: Boost along x^3 -direction.

$$\begin{aligned} \begin{pmatrix} E \\ p^3 \end{pmatrix} &= \exp \left[w \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \begin{pmatrix} m \\ 0 \end{pmatrix} = (\cosh w \mathbb{1} + \sinh w \sigma^1) \begin{pmatrix} m \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} m \cosh w \\ m \sinh w \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
 \text{Boost } \psi: \psi(x) &= \exp\left[-\frac{\omega}{2} \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\right] \sqrt{m} e^{-ip \cdot x} \begin{pmatrix} \xi \\ \xi \end{pmatrix} \\
 &= \left[\cosh \frac{\omega}{2} \mathbb{1} - \sinh \frac{\omega}{2} \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix} \right] \sqrt{m} e^{-ip \cdot x} \begin{pmatrix} \xi \\ \xi \end{pmatrix} \\
 &= \sqrt{m} e^{-ip \cdot x} \begin{pmatrix} \left(\cosh \frac{\omega}{2} \mathbb{1} - \sinh \frac{\omega}{2} \sigma^3\right) \xi \\ \left(\cosh \frac{\omega}{2} \mathbb{1} + \sinh \frac{\omega}{2} \sigma^3\right) \xi \end{pmatrix}
 \end{aligned}$$

$$\text{Use } \cosh \frac{\omega}{2} = \sqrt{\frac{1 + \cosh \omega}{2}}, \quad \sinh \frac{\omega}{2} = \sqrt{\frac{\cosh \omega - 1}{2}},$$

$$\cosh \omega = E/m$$

$$\Rightarrow \sqrt{m} \cosh \frac{\omega}{2} = \sqrt{\frac{m+E}{2}}, \quad \sqrt{m} \sinh \frac{\omega}{2} = \sqrt{\frac{E-m}{2}}$$

$$\psi(x) = \frac{e^{-ip \cdot x}}{\sqrt{2}} \begin{pmatrix} \left(\sqrt{E+m} \mathbb{1} - \sqrt{E-m} \sigma^3\right) \xi \\ \left(\sqrt{E+m} \mathbb{1} + \sqrt{E-m} \sigma^3\right) \xi \end{pmatrix}$$

Example: If $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (spin-up along \hat{x}^3)

$$\psi_+(x) = \frac{e^{-ip \cdot x}}{\sqrt{2}} \begin{pmatrix} \sqrt{E+m} - \sqrt{E-m} \\ 0 \\ \sqrt{E+m} + \sqrt{E-m} \\ 0 \end{pmatrix}$$

$$\begin{aligned}
 \text{Use } \left(\sqrt{E+m} - \sqrt{E-m}\right)^2 &= E+m + E-m - 2\sqrt{E^2-m^2} \\
 &= 2(E-p^3)
 \end{aligned}$$

$$\begin{aligned}
 \text{and } \left(\sqrt{E+m} + \sqrt{E-m}\right)^2 &= E+m + E-m + 2\sqrt{E^2-m^2} \\
 &= 2(E+p^3)
 \end{aligned}$$

$$\rightarrow \Psi_+(x) = \begin{pmatrix} \sqrt{E-p^3} \\ 0 \\ \sqrt{E+p^3} \\ 0 \end{pmatrix} e^{-ip \cdot x}$$

Similarly, with $\vec{s} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

$$\Psi_-(x) = \begin{pmatrix} 0 \\ \sqrt{E+p^3} \\ 0 \\ \sqrt{E-p^3} \end{pmatrix} e^{-ip \cdot x}$$

Ψ_+ and Ψ_- are eigenstates of the helicity operator:

$$h \equiv \hat{\vec{p}} \cdot \frac{\vec{s}}{\hbar} = \frac{1}{2} \hat{p}^i \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$$

$$h \Psi_+ = \frac{1}{2} \Psi_+ \quad , \quad h \Psi_- = -\frac{1}{2} \Psi_-$$

In the massless limit or large energy, $E \approx p^3$

$$\Psi_-(x) \approx e^{-ip \cdot x} \begin{pmatrix} 0 \\ \sqrt{2E} \\ 0 \\ 0 \end{pmatrix}$$

$$\Psi_+(x) \approx e^{-ip \cdot x} \begin{pmatrix} 0 \\ 0 \\ \sqrt{2E} \\ 0 \end{pmatrix}$$

(Now we see the reason for normalizing the solutions w/ the factor of $\sqrt{2E}$ — so the massless limit is well behaved.)

In the massless limit the helicity eigenstates become chirality eigenstates, w/ Ψ_- left-handed and Ψ_+ right-handed.