The positive frequency solutions to the Dirac equation can be written in a form valid for arbitrary 3-momentum:

\[ \psi(x) = u(p)e^{-ipr^+}, \quad \begin{pmatrix} p^0 \\ p^m \end{pmatrix} \cdot \begin{pmatrix} m \\ 0 \end{pmatrix}, \quad p^0 > 0, \]

with two linearly independent solutions for \( u(p) \):

\[
\begin{bmatrix}
\begin{pmatrix} \xi^S \\ \xi^S \end{pmatrix}
\end{bmatrix}^s = \begin{pmatrix} \sqrt{p^m \sigma_m^s} \\ \sqrt{p^m \bar{\sigma}_m^s} \end{pmatrix}, \quad s = 1, 2
\]

\[ \sigma^m = (1, \bar{o})^m, \quad \bar{o}^m = (1, -\bar{o})^m, \]

\[ \xi^1 = (1, 0), \quad \xi^2 = (0, 1) \]

**Normalization:**

\[ u^+(p)u^-(p) = \delta^{rs} p^m \sigma_m^r \xi^r \delta^s_s + \delta^{rs} p^m \bar{\sigma}_m^r \bar{\xi}^r \delta^s_s = 2p^0 \delta^{rs} = 2\omega_p \delta^{rs} \]

where \( \omega_p = \sqrt{p^2 + m^2} \)

Equivalently,

\[ \bar{u}^-(p)u^+(p) = \xi^r \left( p^m \sigma_m^r \right)^{1/2} \left( p^m \bar{\sigma}_m^r \right)^{1/2} \xi^s \]

\[ + \delta^{rs} \left( p^m \sigma_m^r \right)^{1/2} \left( p^m \bar{\sigma}_m^r \right)^{1/2} \bar{\xi}^r \delta^s_s = 2 \xi^r \left( (p^0)^2 - \frac{1}{c^2} \right)^{1/2} \xi^s \]

\[ = 2 \xi^r \left( (p^0)^2 - m^2 \right)^{1/2} \xi^s \]

\[ = 2m \delta^{rs} \]

\[ \Rightarrow \quad \bar{u}^-(p)u^+(p) = 2m \delta^{rs} \]
Negative-Frequency Solutions

\[ \Psi(x) = \phi(p) e^{ip \cdot x}, \quad p^2 p_m = m^2, \quad p^0 > 0 \] (but note positive sign in exponents)

Rest frame: \( p^0 = m, \quad \vec{p} = 0 \).

Dirac equation: \[ - (\gamma_m p_m + m) \phi(p) = 0 \]

\[ -m \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \phi(p) = 0 \]

Solutions: \( \phi(p) = \sqrt{m} \begin{pmatrix} \eta \\ -\eta \end{pmatrix}, \quad \eta = \text{constant} 2\text{-component spinor} \)

In the Weyl basis the Lorentz transformations act independently on the top two and bottom two components, so we can read off the bosonic solutions from the positive-frequency solutions.

\[ \begin{pmatrix} \gamma^5 \phi(p) \\ \gamma^5 \phi(p) \end{pmatrix}, \quad s = 1, 2, \quad \gamma^5 \text{ basis of 2-component spinors.} \]

Normalization: \( \sqrt{r^+(p)} \phi^s(p) = 2 \sqrt{m} \phi^s(p) \)

Equivalently:

\[ \sqrt{r^+(p)} \phi^s(p) = -2 \sqrt{m} \phi^s(p) \]

Furthermore, the positive and negative-frequency solutions are orthogonal in the sense that

\[ \bar{r}^+(p) \bar{r}^+(p) = \bar{r}^-(p) \bar{r}^+(p) = 0 \]