Physical interpretation of Lorentz transformations.

Consider rotations. For a rotation about the $x^2$-axis by an angle $\Theta$ we would write

\[
\begin{pmatrix}
    x^0 \\
    x^1 \\
    x^2 \\
    x^3
\end{pmatrix} \rightarrow \begin{pmatrix} 1 & \cos \Theta & -\sin \Theta \\ -\sin \Theta & \cos \Theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix}
    x^0 \\
    x^1 \\
    x^2 \\
    x^3
\end{pmatrix}
\]

For small angles this becomes, \[
\left(\begin{array}{cc}
1 & -\Theta \\
0 & 1 \\
\end{array}\right) + \Theta(\Theta^2).
\]

We can write the infinitesimal transformation matrix as $\delta^{\mu}_{\nu} + w^{\mu}_{\nu}$, where $w^{\mu}_{\nu}$ is the antisymmetric matrix \[
\left(\begin{array}{ccc}
0 & -\Theta & 0 \\
\Theta & 0 & 0 \\
0 & 0 & 0 \\
\end{array}\right).
\]

For a general rotation by $\Theta$ about the $x^2$ axis we would have for the spatial components $w^{ij}$, \[
w^{ij} = -w^{ji} = w^{ji} = -w^{ji} = \frac{1}{2} \varepsilon^{ijk} \Theta \Theta^k.
\]

For our rotation about $x^2$ we have $w^{12} = \Theta$. 
For a boost in the $x^i$-direction by velocity $v$,

\[
\begin{pmatrix}
  x^0 \\
  x^1 \\
  x^2 \\
  x^3
\end{pmatrix}
\rightarrow
\begin{pmatrix}
  \cosh w & \sinh w \\
  \sinh w & \cosh w
\end{pmatrix}
\begin{pmatrix}
  x^0 \\
  x^1 \\
  x^2 \\
  x^3
\end{pmatrix}
\]

where the boost parameter $w$ satisfies

\[
\cosh w = \sqrt{1 - v^2/c^2}
\]

\[
\tanh w = \frac{v}{c}
\]

For small $\frac{v}{c}$, the transformation matrix becomes,

\[
\begin{pmatrix}
  1 & \frac{v}{c} \\
  \frac{v}{c} & 1
\end{pmatrix} + O \left( \left( \frac{v}{c} \right)^2 \right)
\]

If we again write this as $\delta^{\mu}_{\nu} + w^{\mu}_{\nu}$, then in this example $w^0_1 = w^1_0 = w^{10} = -w^{01} = \frac{v}{c}$.

A general infinitesimal Lorentz transformation is specified by the antisymmetric matrix $w^{\mu \nu}$.

Count # parameters in $w^{\mu \nu}$: \[
\frac{4 \cdot 3}{2} = 6
\]

= 3 rotations + 3 boosts √
We can understand the antisymmetry of \( w_{\mu
u} \) from the defining relation for Lorentz transformations:

\[
\Lambda^m_n = \delta^m_n + w^m_n, \quad \Lambda^m_n \gamma_{\mu \beta} \Lambda^\beta_n = \gamma_{\mu \nu}
\]

\[
(\delta^m_n + w^m_n) \gamma_{\beta} (\delta^\beta_\alpha + w^\beta_\alpha)
\]

\[
= \gamma_{\mu \nu} + \omega_{\alpha \nu} + \omega_{\alpha \mu} + \text{higher order terms} \tag{O(\omega^2)}
\]

(Note that we have been raising and lowering indices with \( \gamma_{\mu \nu} \).)

So to linear order in \( \omega \), \( \omega_{\alpha \nu} = -\omega_{\nu \alpha} \), as promised.
Now back to the Dirac equation. We will introduce a notation that makes the Dirac eqn appear Lorentz invariant, and then we will prove that it is.

\[ i \hbar \frac{\partial \psi}{\partial t} + i \hbar c \bar{\alpha} \cdot \nabla \psi = \beta mc^2 \psi \]

Multiply by \( \beta \), define four "\( \gamma \)-matrices":

\[ \gamma^0 = \beta \]
\[ \gamma^i = \beta \alpha^i \quad , \quad i = 1, 2, 3 \]

Using \( \beta^2 = 1 \):

\[ i \hbar c \frac{\partial}{\partial x^m} \gamma^m \psi = mc^2 \psi \]

This looks covariant, but we should keep in mind that \( \gamma^m \) are just 4 matrices — they don’t transform under Lorentz transformations.

More notation: The Feynman slash, \( \not{p} = \gamma^m p_m \) for any 4-vector \( p^m \).

\[ \not{p} \psi = mc^2 \psi \]

Finally, in "natural units" \( \hbar = c = 1 \)

\[ (i\partial - m) \psi = 0 \]  The nice, compact Dirac eqn.
Two useful representations of the gamma matrices:

**Dirac basis**:
\[ \gamma^0 = \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]
\[ \gamma^i = \beta \alpha^i = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \]

**Weyl basis**:
\[ \gamma^0 = \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]
\[ \gamma^i = \beta \alpha^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \]

**Exercise**: What is \( \alpha^i \) in the Weyl basis?

**Properties of \( \gamma^\mu \)**:
- \( (\gamma^0)^2 = \beta^2 = 1 \)
- \( (\gamma^i)^2 = \beta \alpha^i \beta \alpha^i = -\beta^2 (\alpha^i)^2 = -1 \)
- \( \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 0 \) if \( \mu \neq \nu \)

**Summary**: \[ \{ \gamma^\mu, \gamma^\nu \} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 \gamma^{\mu \nu} \]
We can rederive the Klein-Gordon eqn from the Dirac eqn using our new notation:

\[ i \partial^\mu \phi = m \phi \quad \text{Dirac eqn.} \]
\[ -\partial \partial \psi = i \mu \partial^\mu \psi = m^2 \psi \]
\[ \gamma^m \partial_m \gamma^m \psi = m^2 \psi \]
\[ = -\frac{i}{2} \varepsilon^{mnpq} \partial_m \partial_n \partial_q \psi \]
\[ = -\partial^m \partial_m \phi \]
\[ = -\partial^m \partial_m \psi \]

Hence, \[ -\partial^m \partial_m \psi = m^2 \psi \] Klein-Gordon eqn.

This is the manifestly covariant form of the KG eqn. Both sides of the eqn. transform however \( \phi \) transforms.

To figure out how \( \phi \) transforms we can work backwards — start from the Dirac eqn and figure out what works.

Assume under a Lorentz transformation \( \Lambda \),

\[ \phi(x) \rightarrow S(\Lambda) \phi(\Lambda^{-1} x) \]

for some \( 4 \times 4 \) matrix \( S(\Lambda) \),

Suppose we now transform backwards by \( \Lambda^{-1} \), so we get the sequence of transformations:

\[ \phi(x) \rightarrow S(\Lambda) \phi(\Lambda^{-1} x) \rightarrow S(\Lambda) S(\Lambda^{-1}) \phi(x) \].
But transforming by \( \Lambda \) and then \( \Lambda^{-1} \) should do nothing, so we get

\[
S(\Lambda^{-1}) = S^{-1}(\Lambda)
\]

Now transform the Dirac eqn:

\[
(i \gamma^m \partial_m - m) \Psi(x) \rightarrow [(i \gamma^m (\Lambda^{-1})^\mu_\nu \partial_\nu - m) S(\Lambda^{-1}) \Psi(x)] = 0
\]

Multiply on the left by \( S^{-1}(\Lambda) \):

\[
i S^{-1}(\Lambda) \gamma^m S(\Lambda) (\Lambda^{-1})^\nu_\mu \partial_\nu \Psi - m S^{-1}(\Lambda) \Psi = 0
\]

This has the same form as the original Dirac eqn iff

\[
S^{-1} \gamma^m S(\Lambda^{-1})^\nu_\mu = \gamma^\nu
\]

Given an infinitesimal Lorentz transformation specified by the antisymmetric matrix \( w^{\mu\nu} \) expand \( S(\Lambda) \) in \( w \):

\[
S \approx 1 - \frac{i}{4} \sigma_{\mu\nu} w^{\mu\nu}
\]

\[
S^{-1} \approx 1 + \frac{i}{4} \sigma_{\mu\nu} w^{\mu\nu}
\]

for some set of matrices \( \sigma_{\mu\nu} \), antisymmetric in \( \mu \leftrightarrow \nu \).
\[ S^{-1} \gamma^m S = \Lambda^m_{\nu} \gamma^\nu \]

\[ \Rightarrow \gamma^m + \frac{i}{\hbar} \partial_\alpha \omega^\alpha \gamma^m - \frac{i}{\hbar} \gamma^m \partial_\alpha \omega^\alpha + \delta(m) \]

\[ = \gamma^m + \omega^m_{\nu} \gamma^\nu \]

So we need to find a set of matrices \( \sigma_{\alpha\beta} \) satisfying

\[ \frac{i}{\hbar} \partial_{\alpha\beta} \omega^\alpha \gamma^m - \frac{i}{\hbar} \gamma^m \partial_{\alpha\beta} \omega^\alpha = \omega^m_{\nu} \gamma^\nu \]

The answer: \( \sigma_{\alpha\beta} = \frac{i}{2} [\gamma^\alpha, \gamma^\beta] \)

**Exercise:** Use the antisymmetry of \( \omega^\alpha \) and \( [\gamma^\alpha, \gamma^\beta] = 2 \gamma^\alpha \delta_{\alpha\beta} \) to show that this \( \sigma_{\alpha\beta} \) works.

We have found that the Dirac spinor transforms under Lorentz transformations by the matrix,

\[ S = 1 + \frac{i}{\hbar} [\gamma^m, \gamma^\nu] \omega^\mu_{\nu} \]

In the Dirac representation of the \( \sigma \)-matrices,

\[ \sigma^0i = i \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \sigma^i = \delta^{i0} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \]
Given the infinitesimal Lorentz transformations in the Dirac spinor representation, we can build finite Lorentz transformations.

Recall how this works for rotations:

**Rotate by** \( R_1 \), then \( R_2 \):

\[
\Psi \rightarrow S(R_2)S(R_1)\Psi = S(R_2R_1)\Psi
\]

Rotations are generated by the angular momentum matrices \( J_i \), \( [J_i, J_j] = i \epsilon_{ijk} J_k \).

\( S(R) = \exp(-i \Theta J^i) \) rotates about \( \Theta \) by \( \Theta \).

Any matrices satisfying the algebra \( [J_i, J_j] = i \epsilon_{ijk} J_k \) form a representation.

**Spin-\( \frac{1}{2} \) representation:** \( J^i = \frac{\sigma^i}{2} \).

**Lorentz transformations:**

\[
\Psi \rightarrow S(L_2)S(L_1)\Psi = S(L_2L_1)\Psi
\]

The matrices \( \frac{1}{4} \bar{\sigma} \gamma^\mu \gamma^\nu \) generate Lorentz transformations in the Dirac spinor representation.

\[
S(L) = \exp\left(-\frac{i}{4} \bar{\sigma} \gamma^\mu \gamma^\nu \omega_{\mu\nu}\right)
\]
The Lorentz transformations are parametrized by the antisymmetric matrix \( \omega^{\mu \nu} \).

(To compare with Peskin Ch. 3.2, \( S^{\mu \nu} = \pm 6^{\mu \nu} \)).

Compare with notation of 2-component Pauli spinor in non-relativistic QM:
\[
\phi(x) \rightarrow \exp \left( \frac{i}{\hbar} \omega \cdot \sigma \right) \phi(x')
\]

Spatial rotations: \( \sigma_{ij} \) Hermitian \( \Rightarrow S(\text{rotation}) = \text{unitary} \).

But for boosts, \( \sigma_{0i} \) is not Hermitian:
\[
S = \exp \left( -\frac{i}{\hbar} \omega \cdot \sigma_{01} \right) = \exp \left( -\frac{i}{\hbar} \omega \cdot x \right) = S^+ \neq S^{-1}
\]

So the transformation matrices for boosts are nonunitary. However, \( S^{-1} = S_0 S^+ S_0 \) is valid for both boosts & rotations.

Now consider the Lorentz twist of \( \psi + \gamma \psi \): \( \psi + \gamma \psi \rightarrow \psi + S \gamma + S \gamma \neq \psi + \gamma \psi \)

On the other hand:
\[
\psi + \gamma_0 \psi \rightarrow \psi + S + \gamma_{05} \gamma = \psi + \gamma_0 \gamma \gamma_{05} + \gamma_{05} \gamma
\]
\[
= \psi + \gamma_0 S^{-1} \gamma = \psi + \gamma_0 \psi
\]

So \( \psi + \gamma_0 \psi \) is Lorentz invariant.
How about $\psi \pm \sigma \psi$?

$\psi \pm \sigma \psi \rightarrow \psi \pm s \pm \sigma s \psi$

$\quad = \psi \pm s \sigma s \pm s \sigma s \psi$

$\quad = \psi \pm s \sigma s \pm \sigma s \psi$

$\quad = \psi \pm s \chi \psi$

$\quad = \chi \psi \pm \sigma s \psi$

Hence, $\psi \pm \sigma \psi$ transforms as a $\gamma$-vector.

The combination $\psi \pm \sigma$ appears so often it is given its own notation and a name:

$\bar{\psi} = \psi + \sigma$

Dirac adjoint.

We can also define the Dirac adjoint of an operator:

$\bar{A} = \sigma A + \sigma$

It is defined so that $\bar{\psi} \bar{A} \psi = (\bar{\psi} A \psi)^*$, like the ordinary adjoint.