The Coulomb Potential

In the Center of Mass frame two-body scattering can be interpreted as scattering off of a potential. If we take the non-relativistic limit of a scattering cross section, we can determine the potential by comparison w/ the Born approximation in non-relativistic QM:

The sol'n to the Schrödinger eqn w/ potential $V(\vec{r})$ which looks like a plane wave when $V \to 0$ is approximately

$$\psi(\vec{r}) \approx \frac{1}{(2\pi)^{3/2}}\left[e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r} f(\vec{k},\vec{k}')\right]$$

where $f(\vec{k},\vec{k}') \approx -\frac{m}{2\pi}\int d^3\vec{r}'\, e^{i(\vec{k}-\vec{k}')\cdot\vec{r}'}\, V(\vec{r}')$

$$\equiv -\frac{m}{2\pi}\tilde{V}(\vec{k}-\vec{k}')$$

and $\vec{k}' \equiv |\vec{k}|\,\hat{r}$.

In terms of $f(\vec{k},\vec{k}')$ the differential cross section is,

$$\boxed{\frac{d\sigma}{d\Omega} = |f(\vec{k},\vec{k}')|^2 = \frac{m^2}{4\pi^2}\left|\tilde{V}(\vec{k}-\vec{k}')\right|^2}$$

If $V(\vec{r})$ describes the potential between two particles that are scattering off of one another, then m should be replaced by the reduced mass $\frac{m_1 m_2}{m_1+m_2} = \frac{m}{2}$, if $m_1 = m_2 = m$

The cross section in NRQM should be compared w/ our field theoretic expression,

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 \omega_{Tot}^2} \frac{|\vec{P}_1|}{|\vec{P}_A|} |M|^2$$

$$\approx \frac{1}{64\pi^2 (2m)^2} |M|^2$$

in the non-rel limit of elastic scattering of particles w/ equal mass m.

Comparing the two expressions, we identify

$$\boxed{|\tilde{V}(\vec{k}-\vec{k}')|^2 \approx \frac{1}{16m^4} |M(\vec{k}\to\vec{k}')|^2}$$

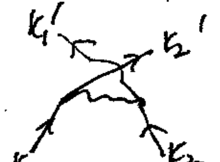
With our signs conventions, comparing NR scattering amplitudes directly gives,

$$\boxed{\tilde{V}(\vec{k}-\vec{k}') \approx -\frac{1}{4m^2} M(\vec{k}\to\vec{k}')}$$

This expression is more useful because it carries information about the <u>sign</u> of the potential.

Now considering electron-electron scattering to lowest order:

$iM =$ 

$+$ 

$$= (-ie)^2 \bar{u}(K_1')\gamma^\mu u(K_1) \frac{-ig_{\mu\nu}}{(K_1-K_1')^2} \bar{u}(K_2')\gamma^\nu u(K_2) - (K_1' \leftrightarrow K_2')$$

Consider the first term in $i\mathcal{M}$.

In the nonrel limit $\bar{u}(k_1')\gamma^0 u(k_1) = u^\dagger(k_1')u(k_1)$
$$\approx 2m\, \xi_1'^\dagger \xi_1$$
where ξ_1 and ξ_1' are the 2-component Pauli spinors describing the initial and final state electron labeled 1.

Also, $\bar{u}(k_1')\gamma^i u(k_1) \to 0$ if $\vec{k}_1, \vec{k}_1' \to 0$.

The nonrelativistic scattering amplitude is,

$$i\mathcal{M} = \frac{ie^2}{-|\vec{k}'-\vec{k}|^2}\left(2m\, \xi_1'^\dagger \xi_1\right)\left(2m\, \xi_2'^\dagger \xi_2\right) g_{00}$$
$$= \frac{-ie^2}{|\vec{k}'-\vec{k}|^2} \cdot 4m^2 \left(\xi_1'^\dagger \xi_1\right)\left(\xi_2'^\dagger \xi_2\right)$$

The spin of each electron is conserved, since
$$\xi_1'^\dagger \xi_1 \equiv \xi^{r_1'\dagger}\xi^{r_1} = \delta^{r_1' r_1}$$
$$\xi_2'^\dagger \xi_2 \equiv \xi^{r_2'\dagger}\xi^{r_2} = \delta^{r_2' r_2}$$

Factoring out the spin-conserving Kronecker δ's and comparing w/ the Born approx, we get the Fourier transformed potential,
$$\boxed{\tilde{V}(\vec{k}-\vec{k}') = \frac{e^2}{|\vec{k}'-\vec{k}|^2}}$$

Fourier transforming we get the potential in coordinate space. We will kill two birds w/ one stone and consider a more general potential $\tilde{V}(\vec{k}) = \frac{e^2}{|\vec{k}|^2 + m^2}$

Then,
$$V(\vec{r}) = e^2 \int \frac{d^3K}{(2\pi)^3} \frac{e^{-i\vec{K}\cdot\vec{r}}}{\vec{K}^2 + m^2}$$

$$= e^2 \int_0^\infty dK \int_0^\pi d\theta \int_0^{2\pi} d\phi \frac{K^2}{(2\pi)^3} \sin\theta \frac{e^{-iK|\vec{r}|\cos\theta}}{K^2 + m^2}$$

$$= \frac{e^2}{(2\pi)^2} \int_0^\infty dK \frac{K^2}{K^2 + m^2} \frac{e^{-iK|\vec{r}|} - e^{iK|\vec{r}|}}{-iK|\vec{r}|}$$

$$= \frac{ie^2}{4\pi^2|\vec{r}|} \int_{-\infty}^{\infty} dK \frac{Ke^{-iK|\vec{r}|}}{K^2 + m^2}$$

Analytically continuing to the complex K-plane and closing the integration contour in the lower half plane,

$$V(\vec{r}) = \frac{ie^2}{4\pi^2|\vec{r}|} (-2\pi i) \frac{(-im)\, e^{-i(-im)|\vec{r}|}}{-2im}$$

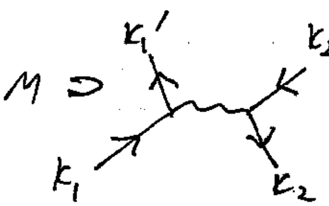
$$\boxed{V(\vec{r}) = \frac{e^2}{4\pi|\vec{r}|} e^{-m|\vec{r}|}}$$ Yukawa potential

As $m \to 0$ we recover the Coulomb potential,

$$\boxed{V(\vec{r}) = \frac{e^2}{4\pi|\vec{r}|}}$$

The sign of the potential teaches us that particles w/ the same charge <u>repel</u> in QED.

What about particle-antiparticle scattering?

$iM \supset$ 

$= (-ie)^2 \bar{u}(k_1')\gamma^\mu u(k_1) \frac{-ig_{\mu\nu}}{(k_1'-k_1)^2} \bar{v}(k_2)\gamma^\nu v(k_2')$

$$\bar{v}(k_2)\gamma^0 v(k_2') = v^\dagger(k_2)v(k_2') \approx 2m\,\xi_2^\dagger \xi_2',$$

just as for particles w/ $v \to u$.

The extra $-$ve sign came from exchanges of fermion fields:

$$\langle k_1', k_2' | : \bar{\psi}\psi \; \bar{\psi}\psi : | k_1, k_2 \rangle$$

$$\propto \langle 0 | b_{k_2'} a_{k_1'} \; \bar{\psi}\psi \; \bar{\psi}\psi \, a^\dagger_{k_1} b^\dagger_{k_2} | 0 \rangle$$

— Requires odd # exchanges of fermion ops to move the ψ's and $\bar{\psi}$'s next to the corresponding creation or annihilation operators $\to$ gives an overall $(-)$.

Hence, the potential between particles and antiparticles takes the same form as the potential betw particles-particles (or antiparticles-antiparticles), but w/ the opposite sign

$\to$ $\boxed{\text{particles and antiparticles attract in QED}}$

Exchange Potentials

In the derivation of the Coulomb potential we quietly ignored one of the contributions to the scattering amplitude,

, which is equal to our previous term w/ $k_1' \leftrightarrow k_2'$, up to a sign.

Hence, we get another term in the potential

$\vec{k}_2' = -\vec{k}_1'$ in COM frame.

$$\tilde{V}_{exch.}(\vec{k}_1 - \vec{k}_2') = \tilde{V}_{exch}(\vec{k}_1 + \vec{k}_1')$$

$$= \frac{-e^2}{|\vec{k}_1 + \vec{k}_1'|^2}$$

We would not have gotten this term had we considered instead scattering of non-identical particles.

This is the exchange potential in the Hamiltonian of non-relativistic QM w/ two identical particles,

$$H = H_0 + \underset{\text{Coulomb Potential}}{V} + \underbrace{VE}_{\text{Exchange Coulomb potential}} \qquad \underset{\text{Exchange operator}}{E}|\vec{r}_1, \vec{r}_2\rangle = \underset{\text{Because these are fermions.}}{-}|\vec{r}_2, \vec{r}_1\rangle$$

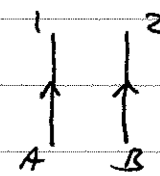
Higher-Order Corrections to Scattering: A First Look

There are a few subtleties that you should be aware of when considering scattering at higher order.

1) Only <u>connected</u> diagrams contribute.

Example: $e^- + e^- \to e^- + e^-$

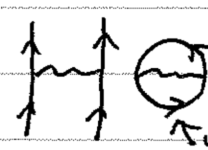
You could draw the Feynman diagram



$$= 4\omega_A \omega_B (2\pi)^6 \delta^3(\vec{p}_1 - \vec{p}_A)\, \delta^3(\vec{p}_2 - \vec{p}_B)$$

but this only contributes to the trivial process in which the final and initial states are identical.

You could also draw diagrams like

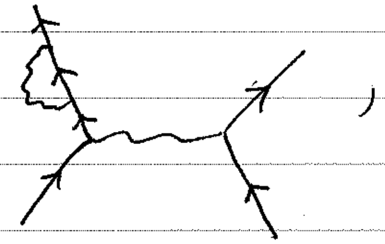


,

"vacuum bubble" – has no external lines

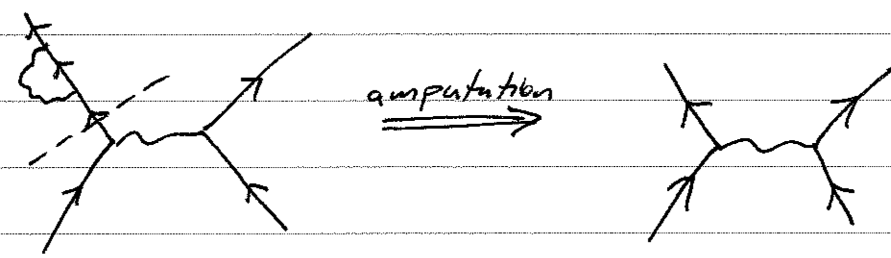
which contain disconnected factors w/ no external lines, but summing over all products of vacuum bubbles just gives an overall phase, corresponding to the inequality between the vacuum energy in the interacting theory and in the free theory.

2) Only <u>amputated</u> diagrams contribute.

You could draw diagrams like  ,

which include modifications to external lines. These terms correspond to the difference between the final (or initial) states in the interacting theory and in the free theory. As long as we use physical quantum numbers in the states (such as mass) we can neglect those terms.

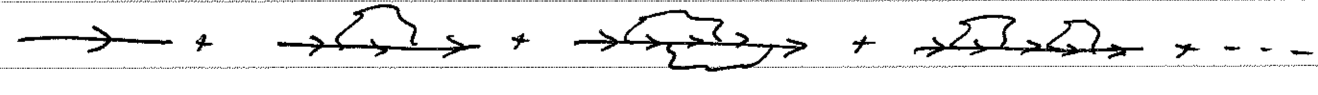
Amputation refers to removal of everything except the most bare of external legs.



3) <u>Soft Bremsstrahlung</u>
No detector in the world can distinguish the state of one electron and the state of one electron and a photon of arbitrarily large wavelength. Hence, to predict cross sections in QED we should sum the cross sections for emission of any number of "soft," i.e. long-wavelength, photons, e.g. 

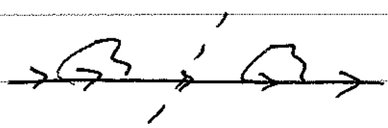
4) Self Energies

If a diagram contains an electron on an *internal* line, i.e. a propagator, then there are infinitely more diagrams that contribute to the same process which consist of modification of the propagator:



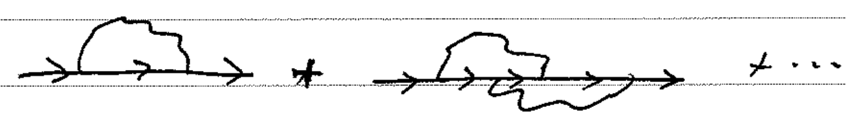
Define a one-particle-irreducible diagram (1PI diagram) to be one in which the diagram can not be separated into disconnected parts by cutting a single internal line.

1PI



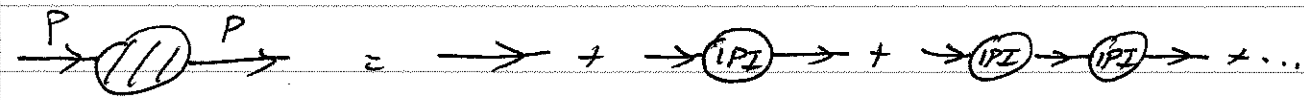
Not 1PI

Define the sum of all 1PI corrections to the modified propagator as $-i\Sigma(p)$.

$-i\Sigma(p) =$ 

$\Sigma(p)$ is called the *electron self energy*.

The sum over all corrections to the electron propagator can be written in terms of the self energy.



All contributions

$$= \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon} + \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}\left(-i\Sigma(p)\right)\frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}$$

$$+ \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}\left[\left(-i\Sigma(p)\right)\frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}\right]^2 + \ldots$$

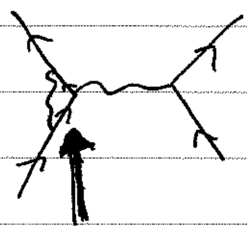
$$= \frac{i}{\not{p}-m+i\epsilon} + \frac{i}{\not{p}-m+i\epsilon}\left[\left(-i\Sigma(p)\right)\frac{i}{\not{p}-m+i\epsilon}\right] + \ldots$$

$$= \frac{i}{\not{p}-m+i\epsilon} \cdot \frac{1}{1-\frac{\Sigma(p)}{\not{p}-m+i\epsilon}} = \boxed{\frac{i}{\not{p}-m-\Sigma(p)+i\epsilon}}$$

☆ The location of the pole in the propagator is typically used to define the <u>mass</u> of the propagating particle. The self-energy corrections can modify the location of the pole, so the mass that appears in the Lagrangian is not necessarily the mass of the corresponding particle.

5) Vertex Corrections

For each vertex in a diagram there are higher-order corrections at the vertex:



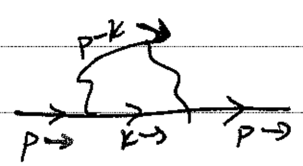
Vertex correction

The vertex correction modifies the form of the electromagnetic interaction. One consequence is the anomalous magnetic moment $g-2 \neq 0$.

gyromagnetic ratio of the electron.

6) Divergences, Regularization and Renormalization

Consider the contribution to the electron self energy at $O(e^2)$:



$$-i\Sigma(p) = \int \frac{d^4k}{(2\pi)^4} (-ie\gamma^\mu) \frac{i(\not{k}+m)}{k^2-m^2+i\epsilon} (-ie\gamma^\nu) \frac{-ig_{\mu\nu}}{(p-k)^2+i\epsilon}$$

Just counting powers of k, it seems that this integral is divergent, and it is.

Somehow these divergences must cancel when asking physical questions, and in QED they do. This is the subject of regularization and renormalization, which you will study next semester.