The Coulomb Potential

In the Center of Mass frame two-body scattering can be interpreted as scattering off of a potential. If we take the non-relativistic limit of a scattering cross section, we can determine the potential by comparison with the Born approximation in non-relativistic QM:

The solution to the Schrödinger equation with potential \( V(\vec{r}) \) which looks like a plane wave when \( V \to 0 \) is approximately

\[
\Psi(\vec{r}) \approx \frac{1}{(2\pi)^{3/2}} \left[ e^{i\vec{p} \cdot \vec{r}} + \frac{e^{i\vec{k} \cdot \vec{r}}}{r} f(\vec{r}, \vec{r}') \right]
\]

where \( f(\vec{r}, \vec{r}') = -\frac{m}{2\pi} \int d^3 \vec{r}'' e^{i(\vec{p} - \vec{k}) \cdot \vec{r}''} \nabla V(\vec{r}'') \)

\[
= -\frac{m}{2\pi} \nabla \nabla \cdot \vec{r} - \vec{r}
\]

and \( \vec{r}' = |\vec{r}| \hat{r} \).

In terms of \( f(\vec{r}, \vec{r}') \) the differential cross section is

\[
\frac{d\sigma}{d\Omega} = |f(\vec{r}, \vec{r}')|^2 = \frac{m^2}{4\pi^2} \left| \nabla V(\vec{r} - \vec{r}') \right|^2
\]

If \( V(\vec{r}) \) describe the potential between two particles that are scattering off of one another, then \( m \) should be replaced by the reduced mass \( \frac{m_1 m_2}{m_1 + m_2} = \frac{m}{2} \) if \( m_1 = m_2 \).
The cross section in NRQM should be compared w/ our field theoretic expression,

\[
\frac{d\sigma}{d\Omega} = \frac{1}{6\pi^2} \frac{\left| \frac{P_i}{E_i} \right|^2}{E_i^2} |M|^2
\]

\[
\approx \frac{1}{6\pi^2 (2m)^2} |M|^2
\]

in the non-rel limit of elastic scattering of particles w/ equal mass \( m \).

Comparing the two expressions, we identify

\[
|\mathcal{V}(E-E')|^2 \approx \frac{1}{16m^4} |M(E\rightarrow E')|^2
\]

With our conventions, comparing NR scattering amplitudes directly gives

\[
\mathcal{V}(E-E') = -\frac{i}{\gamma m^2} M(E\rightarrow E')
\]

This expression is more useful because it carries information about the sign of the potential.

Now considering electron-electron scattering to lowest order:

\[
i M = \begin{array}{c}
\begin{array}{c}
\text{\( k_1' \)}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{\( k_2 \)}
\end{array}
\end{array}
\]
Consider the first term in $iM$.

In the nonrel. limit $\bar{u}(k_1') \delta^0 u(k_1) = u^+(k_1') u(k_1') = 2m \xi_1^+ \xi_1$

where $\xi_1$ and $\xi_1'$ are the 2-component Pauli spinors describing the initial and final state electron labeled 1.

Also, $\bar{u}(k_1') \delta^0 u(k_1) \rightarrow 0$ if $k_1, k_1' \rightarrow 0$.

The nonrel. scattering amplitude is,

\[
iM = \frac{ie^2}{-i k' - i k} (2m \xi_1^+ \xi_1) (2m \xi_2^+ \xi_2) 900
\]

\[
= -\frac{ie^2}{ik^2} \cdot 4m^2 (\xi_1^+ \xi_1) (\xi_2^+ \xi_2)
\]

The spin of each electron is conserved, since

\[
\xi_1^+ \xi_1 = \xi_2^+ \xi_2 = \xi_1' \xi_1
\]

\[
\xi_2^+ \xi_2 = \xi_1'^+ \xi_1' = \xi_2' \xi_2
\]

Factoring out the spin-conserving Kronecker $\delta$'s and comparing with the Born appx, we get the Rainer transformed potential,

\[
\sqrt{V(k-k')} = \frac{e^2}{ik^2 - i k'^2}
\]
Fourier transforming we get the potential in coordinate space. We will kill two birds in one stone and consider a more general potential \( V(\mathbf{r}) = \frac{e^2}{|\mathbf{r}|^2 + m^2} \).

Then \( V(\mathbf{r}) = e^2 \oint \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{e^{-i\mathbf{k} \cdot \mathbf{r}}}{|\mathbf{k}|^2 + m^2} \)

\[
= e^2 \int_0^\infty dk \int_0^\pi d\theta \int_0^{2\pi} d\phi \frac{k^2 \sin \theta}{(2\pi)^3} \frac{e^{-i|\mathbf{k}|r|\cos \theta}}{|\mathbf{k}|^2 + m^2}
\]

\[
= \frac{e^2}{(2\pi)^2} \int_0^\infty dk \frac{k^2}{k^2 + m^2} \frac{e^{-i|\mathbf{k}|r}}{-i|\mathbf{k}|r} - \frac{e^{-i|\mathbf{k}|r}}{-i|\mathbf{k}|r}
\]

\[
= \frac{i e^2}{4\pi^2 |\mathbf{r}|} \int_0^\infty dk \frac{k e^{-i|\mathbf{k}|r}}{k^2 + m^2}
\]

Analytically continuing to the complex \( k \)-plane and closing the integration contour in the lower half-plane,

\[
V(\mathbf{r}) = \frac{i e^2}{4\pi^2 |\mathbf{r}|} (-2\pi i) (-i m) \frac{e^{-i|\mathbf{k}|r}}{i m}
\]

\[
V(\mathbf{r}) = \frac{e^2}{4\pi |\mathbf{r}|} e^{-m|\mathbf{r}|/2m} \text{ Yukawa potential}
\]

As \( m \to 0 \) we recover the Coulomb potential,

\[
V(\mathbf{r}) = \frac{e^2}{4\pi |\mathbf{r}|}
\]
The sign of the potential indicates that particles of the same charge repel in QED.

What about particle-antiparticle scattering?

\[
\langle k_1', k_2' \mid \bar{\psi} \Psi \bar{\psi} \Psi \mid k_1, k_2 \rangle
\]

\[
\rightarrow \text{particles and antiparticles attract in QED}
\]
Exchange Potentials

In the derivation of the Coulomb potential we quietly ignored one of the contributions to the scattering amplitude,

which is equal to our previous term \( \Psi_1 \to \Psi_2 \), up to a sign.

Hence, we get another term in the potential:

\[ \tilde{V}_{\text{exch.}} (\mathbf{r}_1 - \mathbf{r}_2') = \tilde{V}_{\text{exch.}} (\mathbf{r}_1 + \mathbf{r}_2') \]

\[ = \frac{-e^2}{|\mathbf{r}_1 + \mathbf{r}_2'|^2} \]

We would not have gotten this term had we considered instead scattering of non-identical particles.

This is the exchange potential in the Hamiltonian of non-relativistic QM with two identical particles:

\[ H = H_0 + V + V_{\text{exch.}} \]

\[ E|\mathbf{r}_1, \mathbf{r}_2> = -|\mathbf{r}_2, \mathbf{r}_1> \]

Because these are fermions.
Higher-Order Corrections to Scattering:
A First Look

There are a few subtleties that you should be aware of when considering scattering at higher order.

1) Only connected diagrams contribute.

Example: \( e^- + e^- \rightarrow e^- + e^- \)

You could draw the Feynman diagram

\[
\text{Diagram}
\]

\[= \frac{\alpha}{\pi} \frac{A}{B} (2\pi)^4 \delta^4(\vec{p}_1 - \vec{p}_A) \delta^4(\vec{p}_2 - \vec{p}_B)
\]

but this only contributes to the trivial process in which the final and initial states are identical.

You could also draw diagrams like

\[
\text{Diagram with vacuum bubble}
\]

which contain disconnected factors if no external lines, but summing over all products of vacuum bubbles just gives an overall phase, corresponding to the inequality between the vacuum energy in the interacting theory and in the free theory.
2) Only amputated diagrams contribute.

You could draw diagrams like

which include modifications to external lines. These terms correspond to the difference between the final (or initial) states in the interaction theory and in the free theory. As long as we use physical quantum numbers in the states (such as mass) we can neglect those terms.

Amputation refers to removal of everything except the most bare of external legs.

3) Soft Bremsstrahlung

No detector in the world can distinguish the state of one electron and the state of one electron and a photon of arbitrarily large wavelength. Hence, to predict cross sections in QED we should sum the cross sections for emission of any number of "soft," i.e. low-momentum, photons, e.g.
4) **Self Energies**

If a diagram contains an electron on an internal line, i.e. a propagator, then there are infinitely more diagrams that contribute to the same process which consist of modifications of the propagator:

\[
\rightarrow + \rightarrow + \rightarrow + \rightarrow + \cdots
\]

Define a **one-particle irreducible diagram (1PI diagram)** to be one in which the diagram cannot be separated into disconnected parts by cutting a single internal line:

\[
\rightarrow + \rightarrow + \rightarrow \text{ 1PI}
\]

\[
\rightarrow + \rightarrow \not\equiv \text{ 1PI}
\]

Define the sum of all 1PI corrections to the modified propagator as \(-i\Sigma(p)\):

\[
-i\Sigma(p) = \rightarrow + \rightarrow + \cdots
\]

\(\Sigma(p)\) is called the **electron self energy**.

The sum over all corrections to the electron propagator can be written in terms of the self-energy.
\[ P \rightarrow \Gamma \rightarrow \Gamma \rightarrow \Gamma \rightarrow \Gamma \rightarrow \Gamma \rightarrow \cdots \]

All contributing:

\[ \frac{i \, (p^2 + m^2)^{-1}}{p^2 - m^2 + i \epsilon} + \frac{i \, i \, \Sigma(p) \, i \, (p^2 + m^2)^{-1}}{p^2 - m^2 + i \epsilon} \]

\[ + \, \frac{i \, (p^2 + m^2)}{p^2 - m^2 + i \epsilon} \left[ -i \, \Sigma(p) \right] \frac{i}{p^2 - m^2 + i \epsilon} \right] \]

\[ = \frac{i}{p^2 - m^2 + i \epsilon} + \frac{i}{p^2 - m^2 + i \epsilon} \left[ -i \, \Sigma(p) \right] \frac{i}{p^2 - m^2 + i \epsilon} \right] \]

\[ = \frac{i}{p^2 - m^2 + i \epsilon} \times \frac{1}{1 - \frac{\Sigma(p)}{p^2 - m^2 + i \epsilon}} = \frac{i}{p^2 - m - \Sigma(p) + i \epsilon} \]

The location of the pole in the propagator is typically used to define the mass of the propagator particle. The self-energy corrections can modify the location of the pole, so the mass that appears in the Lagrangian is not necessarily the mass of the corresponding particle.
5) **Vertex Corrections**

For each vertex in a diagram there are higher-order corrections at the vertex:

![Diagram of a vertex correction](image)

The vertex correction modifies the form of the electromagnetic interaction. One consequence is the anomalous magnetic moment $g-2 \neq 0$.

The gyromagnetic ratio of the electron.

6) **Divergences, Regularization and Renormalization**

Consider the contribution to the electron self-energy at $O(e^2)$:

![Diagram of electron self-energy](image)

$$
-i \Sigma(p) = \int \frac{d^n\mathbf{k}}{(2\pi)^n} \frac{i(e\mathbf{r} \cdot \mathbf{k})}{k^2 - (ie)^2} \frac{i(k+\mathbf{p})}{(k+\mathbf{p})^2} \left(\frac{-ie^2}{(k^2)^2 + (ie)^2}\right).
$$

Just counting powers of $\mathbf{k}$ it seems that this integral is divergent and it is.
Somehow these divergences must cancel when asking physical questions, and in QED they do. This is the subject of regularization and renormalization, which you will study next semester.