Cross Sections and Decay Rates

The amplitude for an initial state of particles with momenta $p_i$ to evolve into a final state of particles with momenta $p_f$ is $\delta^4(\Sigma p_i - \Sigma p_f)$. Squaring the amplitude gives a factor of $\delta^4(\Sigma p_i - \Sigma p_f)^2$. We need to make sense of this.

In a real experiment each particle is described by a normalizable wavepacket that is localized far from the other particles at early and late times.

A 1-particle state with wavefunction $\psi(x)$ is:

$$|\psi(x)\rangle = \frac{\int d^3x}{(2\pi)^3} \psi(x) |x\rangle$$

(suppressing quantum numbers other than momentum.)

The wavefunctions are normalized as follows:

$$1 = \langle \psi|\psi \rangle = \int \frac{d^3k d^3k'}{(2\pi)^3} \psi^*(k) \psi(k') \langle \vec{k}'|\psi^*|\vec{k}\rangle$$

$$= \int \frac{d^3k d^3k'}{(2\pi)^3} \psi^*(k) \psi(k') \sqrt{2\omega_k} \sqrt{2\omega_{k'}} \langle 0|\bar{q}^+q^0\bar{q}|10\rangle$$

$$= \int \frac{d^3k}{(2\pi)^3} |\psi(k)|^2 = 1$$
We are typically interested in one of two physical situations:

1) A beam of some type of particle collides with a target, which can be fixed or could be another beam of particles. We want to know the transition rate for scattering into some region of final state, \( \frac{\text{per unit time}}{\text{per unit flux}} \) of particles. The **cross section**

\[
\sigma = \frac{\text{# events}}{\text{unit time} \times \text{unit flux}}
\]

The flux is the \( \text{# particles} \) (cross-sectional area \times unit time), so \( \sigma \) has the dimensions of area.

2) A particle, or a collection of particles, decay into lighter particles. We want to know the decay rate per unit time,

\[
\Gamma = \frac{\text{# decays}}{\text{unit time}}
\]

We will consider these two situations separately.

### Cross Sections: Scatter incident wavepackets uniformly distributed in impact parameter \( b \)

\[
\begin{align*}
\vec{B} & \rightarrow \text{(target)} \\
\vec{b} & \uparrow \\
\vec{E} & \rightarrow \\
\vec{B} & \rightarrow \text{(target)} \\
\vec{R} & \rightarrow
\end{align*}
\]
Recall that the momentum operator generates translations so we include a factor of $e^{-i \cdot \mathbf{b} \cdot \mathbf{p}}$ to translate the incident B wavepackets by $\mathbf{b}$ from the collinear wavepacket with $\mathbf{b} = 0$.

$$|\psi_A \psi_B\rangle = \left( \frac{d^3k_A \cdot d^3k_B}{(2\pi)^6 \sqrt{2E_A \cdot 2E_B}} \right) \psi_A(\mathbf{k}_A) \psi_B(\mathbf{k}_B) \ e^{-i \mathbf{b} \cdot \mathbf{p}_B} \ |\mathbf{p}_A \mathbf{p}_B\rangle_{in}$$

We can form wavepackets to describe the out states, but if detectors mainly measure momentum and do not resolve the positions of the final state particles (at least at the level of the de Broglie wavelengths) then it is okay to use out states of definite momenta. (It also makes things a little easier.)

With our relativistically normalized state, the probability for $|\psi_A \psi_B\rangle$ to scatter into $n$ particles with momenta in a region $d^3p_1 \ldots d^3p_n$ is

$$P(A\rightarrow 1, 2, \ldots, n) = \frac{1}{\sigma} \frac{d^3p_f}{(2\pi)^3 2E_f} \ |\langle \omega | p_1 \cdots p_n | \psi_A \psi_B\rangle_{in} |^2$$

For a single target A and many incident particles B with $n_B$ incident particles/unit area evenly distributed in impact parameters $\mathbf{b}$, the # scattering events is

$$N = \sum_{\text{incident \ particles } i} n_i = \int d^2\mathbf{b} \ n_B \ P(\mathbf{b})$$
The number of incident particles/unit area is
\[ n_B = \text{flux} \times \text{time}. \]

The cross section is then
\[ \sigma = \frac{N}{n_B} = \int d^2 \phi \ P(C) \].

If there are \( N_A \) target particles, then the number of events grows with \( N_A \). The cross section is defined to be invariant w/r/t \( N_A \).

For example, consider two bunches of particles from a beam of length \( l_B \) and \( l_A \), and densities \( \rho_B \) and \( \rho_A \), cross-sectional area \( A \):

\[ \rho = \text{# particles/unit volume} \]
\[ n_B = \rho_B l_B , \quad n_A = \rho_A l_A \]
\[ N_A = \rho_A l_A \cdot A \]

Then
\[ \sigma = \frac{\# \text{events}}{n_B \cdot N_A}. \]

The differential cross section for scattering into a region of momentum \( \frac{T}{f} d^2 p_f \) is:
\[ d\sigma = \frac{T}{f} \frac{d^2 p_f}{(2\pi)^2 \lambda^2} \int d^2 b \left| \langle p \cdots p_n | u_A u_B (b) \rangle \right|^2. \]
In terms of the scattering matrix $S$,

$$d\sigma = \frac{1}{16} \frac{d^3p_f}{(2\pi)^3} \int d^2b \left( \frac{d^3k_A}{(2\pi)^3} \frac{d^3k_B}{(2\pi)^3} \right) \frac{d^3k'_A}{(2\pi)^3} \frac{d^3k'_B}{(2\pi)^3} \frac{d^3k_A'}{(2\pi)^3} \frac{d^3k_B'}{(2\pi)^3}$$

$$\langle \psi_A (k_A) \psi_B (k_B) \psi_A^* (k'_A) \psi_B^* (k'_B) \rangle \ e^{-i \mathbf{b} \cdot (\mathbf{p}_B - \mathbf{p}_B')}$$

$$\langle p_{1\ldots n} | (S^-1) | k_A k_B \rangle \langle p_{1\ldots n} | (S^-1) | k_A' k_B' \rangle^*$$

Define the invariant scattering amplitude $M(k_A k_B \rightarrow p_{1\ldots n})$ by factoring out the momentum-conserving delta function from the $S$-matrix element:

$$\langle p_{1\ldots n} | (S^-1) | k_A k_B \rangle = i \cdot M(k_A k_B \rightarrow p_{1\ldots n}) (2\pi)^4 \delta^4(k_A + k_B - 2p_f)$$

Also use $\int d^2b \ e^{-i \mathbf{b} \cdot (\mathbf{p}_B - \mathbf{p}_B')} = (2\pi)^2 \delta^2(k_B - k_B')$

Consider the factor,

$$\frac{d^3k_A'}{(2\pi)^3} \frac{d^3k_B'}{(2\pi)^3}$$

$$\frac{1}{(2\pi)^6 \sqrt{2w_A' 2w_B'}}$$

$$\langle \psi_A (k_A') \psi_B (k_B') \psi_A^* (k'_A) \psi_B^* (k'_B) \rangle (2\pi)^4 \delta^4(k_A + k_B - 2p_f)$$

$$\times \delta^2(k_B - k_B')$$

$$\int d^2b' \ e^{-i \mathbf{b}' \cdot (\mathbf{p}_B - \mathbf{p}_B')} = (2\pi)^2 \delta^2(k_B - k_B')$$

$$\times \delta^2(k_B - k_B')$$

$$\int d^3k_{1A}' \frac{d^3k_{1B}'}{(2\pi)^6 \sqrt{2w_{1A}' 2w_{1B}'}}$$

$$\langle \psi_A (k_{1A}') \psi_B (k_{1B}') \psi_A^* (k_{1A}') \psi_B^* (k_{1B}') \rangle (2\pi)^4 \delta^4(k_{1A} + k_{1B} - 2p_f)$$

$$\times \delta^2(k_{1B} - k_{1B}')$$

$$\times \delta^2(k_{1B} - k_{1B}')$$
\[
\frac{d^3 K_A}{(2\pi)^3} \frac{d^3 K_B}{(2\pi)^3} \left| \Phi_A(K_A)^2 \left[ \Phi_B(K_B) \right]^2 \right| \frac{M(P_A \rightarrow P_1 P_2 \cdots P_n)}{2w_A 2w_B |V_A - V_B|} \]

We have assumed that the wavepackets are localized around \( \hat{P}_A \) and \( \hat{P}_B \).

If the detector cannot resolve the spread in momentum due to the fact that \( \Phi_A(K) \) and \( \Phi_B(K) \) are not \( \delta \)-functions, then we can replace \( (K_A, K_B) \) by \( (\hat{P}_A, \hat{P}_B) \) in the \( \delta^4(K_A+K_B - \hat{P}_A) \).

We then obtain our final expression for the differential cross-section:
\[ d\sigma = \frac{|M(p_A, p_B \rightarrow p^- - p_0)|^2}{2\omega_A \cdot 2\omega_B \cdot |\tilde{v}_A - \tilde{v}_B|} \left( \frac{1}{\sqrt{g}} \frac{d^3p_f}{(2\pi)^3 2\omega_f} \right) (2\pi)^4 \delta^4(p_A + p_B - E p_f) \]

\[ D_n = n \text{-body invariant phase space density} \]

Note that \( \omega_A \cdot \omega_B \cdot |\tilde{v}_A - \tilde{v}_B| = |\omega_B \cdot p_A^2 - \omega_A \cdot p_B^2| = |E_{x+y} \cdot p_A^m \cdot p_B^n| \)

is Lorentz invariant under boosts in the \( z \)-direction.

Hence, it is the same in the lab frame and the CM frame.

**\( D_2 \): 2-body final state in CM frame**

\[ D_2 = \frac{d^3p_1 \cdot d^3p_2}{(2\pi)^6 2\omega_1 \cdot 2\omega_2} \]

\[ = \frac{d^3p_1}{(2\pi)^3 \cdot 4\omega_1 \omega_2} \quad 2\pi \delta(w_1 + w_2 - w_{\text{tot}}) \bigg|_{\vec{p}_2 = -\vec{p}_1} \]

Use \( w_1^2 = \vec{p}_1^2 + m_1^2 \), \( w_2^2 = \vec{p}_2^2 + m_2^2 = \vec{p}_1^2 + m_2^2 \)

Define \( \vec{p}_i \equiv |\vec{p}_i| \rightarrow \delta(w_1 + w_2 - w_{\text{tot}}) \frac{\partial \omega(w_1 + w_2)}{\partial \vec{P}_1} \bigg|_{\vec{p}_1 = \vec{p}_2} \frac{\vec{p}_1}{\omega_1} \frac{1}{w_1 \cdot w_2} \cdot w_{\text{tot}} \)

\[ \frac{\partial \omega(w_1 + w_2)}{\partial \vec{P}_1} = \frac{\vec{p}_1}{\omega_1} + \frac{\vec{p}_1}{\omega_2} = \frac{\vec{p}_1}{w_1 \cdot w_2} \frac{w_{\text{tot}}}{w_1 \cdot w_2} = \frac{\vec{p}_1 \cdot w_{\text{tot}}}{w_1 \cdot w_2} \]

Also use \( d^3p_1 = d\vec{p}_1 \cdot d\xi_1 \). Then,

\[ D_2 = \frac{1}{16\pi^2} \frac{\vec{p}_1 \cdot d\xi_1}{w_{\text{tot}}} \]
The differential cross section \( \frac{d\sigma}{d\Omega} \) for a 2-body final state is given by:

\[
\frac{d\sigma}{d\Omega} = \frac{1}{4\mu_A \mu_B} \frac{|M|^2}{|\vec{\nu}_A - \vec{\nu}_B|} \cdot \frac{p_i}{16 \pi^2 \omega_{\text{tot}}}.
\]

In the center-of-mass frame:

\[
\omega_A \omega_B |\vec{\nu}_A - \vec{\nu}_B| = |\omega_B \vec{p}_A - \omega_A \vec{p}_B| = |\omega_B \vec{p}_A + \omega_A \vec{p}_A| = \omega_{\text{tot}} p_A
\]

So, final momentum \( p_i = |\vec{p}_i| \)

\[
\frac{d\sigma}{d\Omega} = \frac{1}{64 \pi^2 \omega_{\text{tot}}} \frac{p_i}{p_A} |M|^2
\]

initial momentum \( p_A = |\vec{p}_A| \)

Note that \( \frac{d\sigma}{d\Omega} \to \infty \) if \( p_A \to 0 \) and \( p_i \neq 0 \).

As \( p_A \to 0 \), the amount of time the two particles remain near one another grows, so there's more time for scattering to occur.

Example: \( e^+ e^- \to \mu^+ \mu^- \)

The Lagrangian for QED coupled to electrons and muons is:

\[
L = \bar{\psi}_e (i \gamma^\mu - m_e - eA) \psi_e + \bar{\psi}_\mu (i \gamma^\mu - m_\mu - eA) \psi_\mu - \frac{i}{4} F_{\mu\nu}^2
\]

where \( \psi_e \) is the electron field, and \( \psi_\mu \) is the muon field.
There are two vertices now: \[ \overline{e} \gamma_{
u} (e^- \gamma^\nu) \] \[ e \to \frac{i(k + me)}{k^2 - m_e^2 + i\epsilon} \]

And two fermion propagators:

\[ \overline{m} \gamma_{\mu} (m^\mu \gamma^\nu \nu) \]

\[ m \to \frac{i(k + m_m)}{k^2 - m_m^2 + i\epsilon} \]

Because the electron and muon are distinguishable, there is only one Feynman diagram that contributes to \( e^+ e^- \to \mu^+ \mu^- \):

\[
\sum_{\text{spins}} |M|^2 = \sum_{\text{spins}} \frac{8\epsilon^4}{(p_4 + p_B)^4} \left[ (p_4 \cdot p_i)(p_B \cdot p_i) + (p_4 \cdot p_B)(p_B \cdot p_i) \right] + \frac{m_m^2}{m_m^2} (p_4 \cdot p_B)
\]

where we have factored out the \((2\pi)^4 \delta^4(p_4 + p_B - p_1 - p_2)\).

Squaring the amplitude and summing over initial spins, summing over final spins gives:

\[
\frac{1}{4} \sum_{\text{spins}} |M|^2 = \frac{8\epsilon^4}{(p_4 + p_B)^4} \left[ (p_4 \cdot p_i)(p_B \cdot p_i) + (p_4 \cdot p_B)(p_B \cdot p_i) \right] + \frac{m_m^2}{m_m^2} (p_4 \cdot p_B)
\]

when we have approximated \(m_e = 0\) because \(\frac{m_e}{m_m} \approx \frac{1}{200}\).
Kinematics: Center-of-Mass Frame

\[ P_4 = (E_4, E_4) \quad \rightarrow \quad P_1 = (E_1, E_1) \]

\[ P_B = (E_1 - E_2) \quad \Rightarrow \quad \theta \]

\[ P_4 \rightarrow \theta \quad P_B = (E_1 - E_2) \quad |P_1| = \sqrt{E^2 - m_1^2} \]

\[ P_1 \cdot \hat{z} = |P_1| \cos \theta \]

To compute the differential cross section, we need to express kinematic factors in terms of \( E \) and \( \theta \):

\[ (P_4 + P_3)^2 = 4E^2, \quad (P_4 \cdot P_3) = 2E^2, \quad E_{tot} = 2E \]

\[ (P_4 \cdot P_1) = (P_3 \cdot P_1) = E^2 - E_1P_1 \cos \theta \]

\[ (P_4 \cdot P_2) = (P_3 \cdot P_2) = E^2 + E_1P_1 \cos \theta \]

Then \[ \frac{1}{4} \sum_{\text{spin}} |M|^2 = \frac{8\pi^4}{16E^4} \left[ E^2(E - |P_1| \cos \theta)^2 + E^2(E + |P_1| \cos \theta)^2 + 2E^2 \right] \]

\[ = \frac{e^4}{2E^2} \left[ 2(E^4 + m_1^2E^2) + 2|P_1|^2 \cos^2 \theta \right] \]

\[ = e^4 \left[ 1 + \frac{m_1^2}{E^2} \right] + \left( 1 - \frac{m_1^2}{E^2} \right) \cos^2 \theta \]

Since we are treating the electrons as massless, we also have \( |\vec{P}_4 - \vec{P}_3| = 2 \). Putting it together,

\[ \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \frac{\sqrt{E^2 - m_1^2}}{E} e^4 \left[ (1 + \frac{m_1^2}{E^2}) + (1 - \frac{m_1^2}{E^2}) \cos^2 \theta \right] \]

\[ \left( \alpha = \frac{e^2}{4\pi} \right) \]

\[ = \frac{e^2}{16E^2} \left[ (1 + \frac{m_1^2}{E^2}) + (1 - \frac{m_1^2}{E^2}) \cos^2 \theta \right] \]
Integrating over the scattering angle $\Theta$ gives the total X-section:

$$\sigma_{\text{Tot}} = \frac{\alpha^2}{16 E^2} \sqrt{1 - \frac{m_n^2}{E^2}} \int_0^{2\pi} d\phi \left[ \left(1 + \frac{m_m^2}{E^2}\right) + \left(1 - \frac{m_m^2}{E^2}\right) \cos^2\Theta \right]$$

$$= \frac{\alpha^2}{16 E^2} \sqrt{1 - \frac{m_n^2}{E^2}} \left[ \left(1 + \frac{m_m^2}{E^2}\right) \cdot 4\pi + \frac{4\pi}{3} \left(1 - \frac{m_m^2}{E^2}\right) \right]$$

$$= \frac{4\pi \alpha^2}{48 E^2} \sqrt{1 - \frac{m_n^2}{E^2}} \left[ 4 + 2 \frac{m_m^2}{E^2} \right]$$

There is a lot more to say about the $e^+e^- \rightarrow \mu^+\mu^-$ cross-section, but we're short on time so we'll move on.

2) Decay Rates: Consider a 1-particle wavepacket centered about the rest frame $p^{(i)}_0 = (m, \vec{0})$.

$$\left| e \right>_m^i = \sum \frac{d^3 k}{(2\pi)^3 \sqrt{2} \omega_k} \left| e \left( \vec{E} \right) \right> \left| \vec{P}^i_m \right>$$,

where

$$\int \frac{d^3 k}{(2\pi)^3} \left| e \left( \vec{E} \right) \right|^2 = 1, \quad \left< e \right| e \left| e \right> = 1.$$  

The probability of decaying into a final state with particles around momenta $\vec{P}_1, \vec{P}_2, ..., \vec{P}_n$ is

$$P(e \rightarrow 1, 2, ..., n) = \prod_i \frac{d^3 p_i}{(2\pi)^3 2\omega_{p_i}} \left| \left< \vec{P}_1, \vec{P}_2, ..., \vec{P}_n \right| e \right>^2,$$
\[ P(\ell \to 1 \cdots n) = \frac{\prod d^3 P_F}{f} \frac{d^3 P_F}{(2\pi)^3 2\omega_F} \left| \langle p_{i-n} | (S-1) | \ell \rangle \right|^2 \]
\[ = \frac{\prod d^3 P_F}{f} \frac{d^3 P_F}{(2\pi)^3 2\omega_F} \int \frac{d^3 k \, d^3 k'}{(2\pi)^6 \sqrt{2\omega_k 2\omega_{k'}}} \left\{ \phi(k) \phi(k') \langle p_{i-n} | (S-1) | \ell \rangle \right\} \times \left\{ \langle p_{i-n} | (S-1) | \ell' \rangle \right\} \]
\[ = \frac{\prod d^3 P_F}{f} \frac{d^3 P_F}{(2\pi)^3 2\omega_F} \int \frac{d^3 k \, d^3 k'}{(2\pi)^6 \sqrt{2\omega_k 2\omega_{k'}}} \left\{ \phi(k) \phi(k') \right\} \times iM(k \to p_{i-n}) (2\pi)^4 \delta^4(k' - k_F) \times (-iM(k' \to p_{i-n})^*) (2\pi)^4 \delta^4(k' - k_F') \]
\[ \propto \frac{\prod d^3 P_F}{f} \frac{1}{(2\pi)^3 2\omega_F} \frac{|M(P_0 \to p_{i-n})|^2}{2\omega_F} \left(2\pi\right)^4 \delta^4(P_{00} - \Sigma p_F) \times \left(2\pi\right)^4 \delta^4(w_0 - \Sigma \omega_F) \int \frac{d^3 k}{(2\pi)^3} \left| \phi(k) \right|^2 \]
\[ = \frac{\prod d^3 P_F}{f} \frac{1}{(2\pi)^3 2\omega_F} \frac{|M|^2}{2m} \left(2\pi\right)^3 \delta^3(P_{00} - \Sigma p_F) \left(2\pi\right)^2 \delta^4(w_0 - \Sigma \omega_F)^2. \]

We got another \( \delta \)-fn squared. If a particle is unstable and decays we can't really make an in-state in the infinite past. But it does make sense to consider the probability of decay per unit time.

We can write \( (2\pi)^2 \left( \delta^4(w_0 - \Sigma \omega_F) \right)^2 = \int dt \int dt' e^{i(w_0 - \Sigma \omega_F)(t+t')} \)
\[ = (2\pi) \delta^4(w_0 - \Sigma \omega_F) \int dt \]
\[ = (2\pi T) \delta^4(w_0 - \Sigma \omega_F) \]

We now divide by this factor of \( T \).
The differential decay rate is:

$$d \Gamma = \frac{P(e \to p_1 \ldots p_n)}{T} = \frac{1}{2m} \frac{1}{f_1 \ldots f_n} \frac{d^3 p_i}{(2\pi)^4} \delta^4(p_{10} - \Sigma p_i) \times |M(e \to p_1 \ldots p_n)|^2$$

In terms of the $n$-body invariant phase space factor $D_n$,

$$d \Gamma = \frac{1}{2m} |M|^2 D_n.$$

The total decay rate is $\Gamma = \int d\Gamma$.

Example: Three real scalars $A, B, C$ with coupling $L_I = -g ABC$.

Assume $m_A > m_B + m_C$ so that the decay $A \to B + C$ is kinematically allowed.

The lowest order invariant amplitude is

![Diagram]

\[ iM = -ig. \]

Using our previous expression for $D_2$,

$$d \Gamma = \frac{1}{2m_A} \cdot g^2 \cdot \frac{1}{16 \pi^2} \frac{|\beta|^2 dR}{m_A}$$

where $|\beta|$ is determined by

$$m_A = \sqrt{\beta^2 m_B^2 + \sqrt{1 \beta^2 + m_C^2}}$$

Total width:

$$\Gamma = \int d\Gamma = \frac{g^2 |\beta|}{8\pi m_A^2}$$