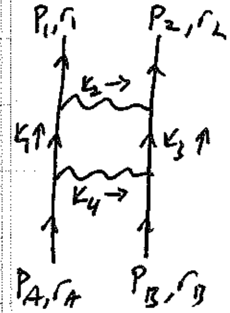
Loop Integrals

In tree level diagrams all internal 4-momenta are determined by momentum conserving δ-functions at each vertex. In general, however, not all internal momenta will be determined by the δ-fns. In that case there will be integrals $\int \frac{d^4k}{(2\pi)^4}$ for each undetermined momentum.
Undetermined momenta always come from closed loops, and are called loop momenta.

Example: At $O(e^4)$ in $e^- + e^- \to e^- + e^-$:



The 4-momentum conserving δ-fns give:

$$\left.\begin{aligned} p_A - k_1 - k_4 &= 0 \\ p_B - k_3 + k_4 &= 0 \\ -p_1 - k_2 + k_1 &= 0 \\ -p_2 + k_3 + k_2 &= 0 \end{aligned}\right\}$$

Summing these gives $p_A + p_B - p_1 - p_2 = 0$, which contains no k_i's.

There is one undetermined momentum that is integrated over. In terms of $k_2 \equiv k$, $k_3 = p_2 - k$, $k_4 = k_3 - p_B = p_2 - p_B - k$, $k_1 = p_1 + k$.
The diagram evaluates to:

$$(-ie)^4 \int \frac{d^4k}{(2\pi)^4} \bar{u}^{r_1}(p_1) \gamma^\mu \frac{i(\not{p}_1 + \not{k} + m)}{(p_1+k)^2 - m^2 + i\epsilon} \gamma^\lambda u^{r_A}(p_A) \; \bar{u}^{r_2}(p_2) \gamma^\nu \frac{i(\not{p}_2 - \not{k} + m)}{(p_2 - k)^2 - m^2 + i\epsilon} \gamma^\sigma u^{r_B}(p_B)$$

$$\times \frac{(-i g_{\mu\nu})}{k^2 + i\epsilon} \frac{(-i g_{\lambda\sigma})}{(p_2 - p_B - k)^2 + i\epsilon} \times (2\pi)^4 \delta^4(p_A + p_B - p_1 - p_2)$$

Fermion Loops: While there is no general rule for the overall sign of a diagram, there is one consistent source of minus signs — Every closed fermion loop gives a (-1) times the trace of a product of Dirac propagators and γ-matrices.

Example:

contains $\bar\psi_1\gamma^\mu\psi_1\;\bar\psi_2\gamma^\nu\psi_2\;\bar\psi_3\gamma^\lambda\psi_3\;\bar\psi_4\gamma^\sigma\psi_4$

$$= \mathrm{Tr}\;\bar\psi_1\gamma^\mu\psi_1\,\bar\psi_2\gamma^\nu\psi_2\,\bar\psi_3\gamma^\lambda\psi_3\,\bar\psi_4\gamma^\sigma\psi_4$$

$$= -\,\mathrm{Tr}\;\psi_4\bar\psi_1\gamma^\mu\,\psi_1\bar\psi_2\,\gamma^\nu\,\psi_2\bar\psi_3\,\gamma^\lambda\,\psi_3\bar\psi_4\,\gamma^\sigma$$

where $\psi_1 \equiv \psi(x_1)$, etc.

We now have all of the Feynman rules for the evaluation of any contribution to any matrix element of the S-matrix to any order in perturbation theory, expressed in terms of integrals over loop momenta.

Next semester we will develop some standard techniques for evaluating most of the loop integrals.

Squaring the Amplitude: Spin Averages and Spin Sums

The probability to observe a certain scattering process depends on the <u>squared magnitude</u> of the corresponding scattering amplitude. Depending on the setup of a particular scattering experiment, it may also be necessary to sum over a collection of squared amplitudes.

Consider the scattering of electrons and/or positrons. The initial state may be prepared in an eigenstate of helicity, or more commonly a beam of such fermions. In that case the Dirac spinors $u^{(r)}(p)$ are determined by the momentum p and spin r, and the products of Dirac spinors and γ-matrices is simply evaluated, and its magnitude squared to determine the probability of scattering into a particular state. Beams of fermions with definite helicity/spin are called <u>polarized</u>.

Often, fermions are created with a known momentum, but the spins are <u>unpolarized</u>, i.e. equally distributed. In that case, the probability of scattering a particular set of fermions should be <u>averaged over initial spins.</u>

Also, if the detectors in the experiment only measure momentum but don't care about spin, then the probability to detect a set of fermions should be <u>summed over final spins.</u>

Casimir's Trace Trick:

We will now develop some technology for performing the sums and averages over spins. These techniques extend beyond QED, so we will try to keep their context quite general.

Consider a scattering amplitude which includes a factor of the form $\bar{u}^{r_a}(p_a)\,\Gamma_1\, u^{r_b}(p_b) \equiv \bar{u}(a)\,\Gamma_1\, u(b)$, where Γ_1 is some 4×4 matrix.

The squared amplitude may include a factor of the form $[\bar{u}(a)\Gamma_1 u(b)][\bar{u}(a)\Gamma_2 u(b)]^*$.

For example, in QED Γ_1 might be γ^μ and Γ_2 might be γ^ν.

$$[\bar{u}(a)\Gamma_1 u(b)][\bar{u}(a)\Gamma_2 u(b)]^*$$

$$= [\bar{u}(a)\Gamma_1 u(b)][u(b)^\dagger \Gamma_2^\dagger \gamma^{0\dagger} u(a)]$$

$$= [\bar{u}(a)\Gamma_1 u(b)][u(b)^\dagger \gamma^0\gamma^0 \Gamma_2^\dagger \gamma^0 u(a)]$$

$$= [\bar{u}(a)\Gamma_1 u(b)][\bar{u}(b)\overline{\Gamma_2} u(a)], \text{ where } \overline{\Gamma_2} = \gamma^0\Gamma_2^\dagger\gamma^0.$$

Suppose $u(a)$ describes a final state electron, and $u(b)$ describes an initial state electron. Summing over a-spins and averaging over b-spins requires us to evaluate $\frac{1}{2}\sum_{r_a}\sum_{r_b} \bar{u}^{r_a}(p_a)\Gamma_1 u^{r_b}(p_b)\, \bar{u}^{r_b}(p_b)\,\overline{\Gamma_2}\, u^{r_a}(p_a)$.

↑ from averaging.

Each term in the sum is a number, ie has no spinor matrix structure, so it is equal to its trace. Using cyclicity of the trace (write out all the spinor indices to convince yourself that this works):

$$\boxed{\sum_{a\text{ spins}} \sum_{b\text{ spins}} \left[\bar{u}(a)\Gamma_1 u(b)\right]\left[\bar{u}(a)\Gamma_2 u(b)\right]^*}$$

$$= \sum_{a\text{ spins}} \sum_{b\text{ spins}} \left[\bar{u}(a)\Gamma_1 u(b)\, \bar{u}(b)\, \bar{\Gamma}_2\, u(a)\right]$$

$$= \sum_{a\text{ spins}} \sum_{b\text{ spins}} \mathrm{Tr}\left[u(a)\bar{u}(a)\,\Gamma_1\, u(b)\bar{u}(b)\,\bar{\Gamma}_2\right]$$

$$\boxed{= \mathrm{Tr}\left[(\not{p}_a + m_a)\Gamma_1(\not{p}_b + m_b)\bar{\Gamma}_2\right]}$$

where in the last step we used $\sum_r u^r(p)\bar{u}^r(p) = \not{p} + m$.

Similarly, if the scattering amplitude involved v's instead of u's,

$$\boxed{\begin{aligned}&\sum_{a\text{ spins}} \sum_{b\text{ spins}} \left[\bar{v}(a)\Gamma_1 v(b)\right]\left[\bar{v}(a)\Gamma_2 v(b)\right]^* \\ &\quad = \mathrm{Tr}\left[(\not{p}_a - m_a)\Gamma_1(\not{p}_b - m_b)\bar{\Gamma}_2\right]\end{aligned}}$$

, and so on.

Examples of Dirac Adjoints:

$$\Gamma_2 = \gamma^\mu \rightarrow \bar{\Gamma}_2 = \gamma^0 \gamma^{\mu\dagger} \gamma^0 = \gamma^\mu$$

$$\Gamma_2 = i\gamma^5 \rightarrow \bar{\Gamma}_2 = \gamma^0 (i\gamma^5)^\dagger \gamma^0 = i\gamma^5$$

We already know how to evaluate traces of products of γ-matrices, so we're in good shape.

Example: $\Gamma_1 = 1$, $\Gamma_2 = 1 \rightarrow \overline{\Gamma}_2 = \gamma^0\gamma^0 = 1$, $m_a = m_b \equiv m$

$$\sum_{a \text{ spins}} \sum_{b \text{ spins}} [\bar{u}(a)u(b)][\bar{u}(a)u(b)]^*$$

$$= \sum_{a \text{ spins}} \sum_{b \text{ spins}} \mathrm{Tr}[u(a)\bar{u}(a)u(b)\bar{u}(b)]$$

$$= \mathrm{Tr}[(\not{p}_a + m)(\not{p}_b + m)]$$

$$= \mathrm{Tr}[\not{p}_a\not{p}_b + m(\not{p}_a + \not{p}_b) + m^2]$$

$$= 4p_a \cdot p_b + 0 + 4m^2$$

$$= 4(p_a \cdot p_b + m^2)$$

Summary of Trace Formulae

$\mathrm{Tr}\, \mathbb{1} = 4$

$\mathrm{Tr}\,(\text{odd } \# \ \gamma\text{'s}) = 0$

$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$

$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$

$\mathrm{Tr}\, \gamma^5 = 0$

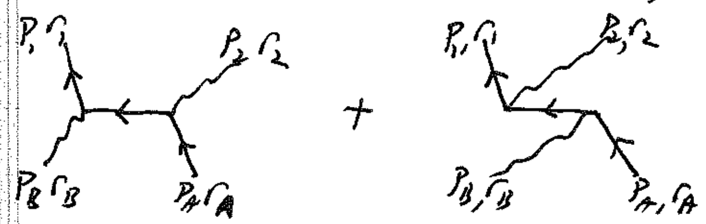
$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^5) = 0$

$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) = -4i\epsilon^{\mu\nu\rho\sigma}$

Photon Polarization Sums and Averages

Beams of light may also be either polarized or unpolarized. If a beam of light is unpolarized, its corresponding helicity in the squared scattering amplitude should be averaged over. If a detector does not measure a photon's helicity, then the final state helicities should be summed over.

Consider Compton Scattering to $\mathcal{O}(e^2)$:



$$= (2\pi)^4 \delta^4(p_1 + p_2 - p_A - p_B)\,(-ie)^2\, i\, \varepsilon_\mu^{(r_2)}(p_2)^* \varepsilon_\nu^{(r_B)}(p_B)$$

$$\bar{u}^{r_1}(p_1)\left[\frac{\gamma^\mu(\not{p}_A + \not{p}_B + m)\gamma^\nu}{(p_A + p_B)^2 - m^2 + i\epsilon} + \frac{\gamma^\nu(\not{p}_A - \not{p}_2 + m)\gamma^\mu}{(p_A - p_2)^2 - m^2 + i\epsilon}\right] u^{r_A}(p_A)$$

Let's simplify this a bit. Use $p_A{}^2 = p_1{}^2 = m^2$
$p_B{}^2 = p_2{}^2 = 0$.

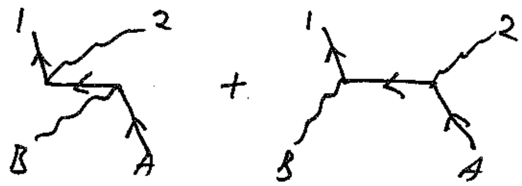
$$\left.\begin{aligned} \rightarrow (p_A + p_B)^2 - m^2 &= 2 p_A \cdot p_B \\ (p_A - p_2)^2 - m^2 &= -2 p_A \cdot p_2 \end{aligned}\right\} \text{Denominators.}$$

Since there are no loop integrals, the $i\epsilon$'s are unimportant

To simplify the numerators we can use some Dirac algebra:

$$(\not{p}_A + m)\gamma^\nu u^{r_A}(p_A) = (2p_A^\nu - \gamma^\nu \not{p}_A + \gamma^\nu m) u^{r_A}(p_A)$$
$$= 2p_A^\nu u^{r_A}(p_A) - \gamma^\nu \underbrace{(\not{p}_A - m) u^{r_A}(p_A)}_{0 \text{ by Dirac eqn}}$$
$$= 2p_A^\nu u^{r_A}(p_A)$$

We now obtain:



$$= -ie^2 \epsilon_\mu(p_2)^* \epsilon_\nu(p_B)\, \bar{u}^{r_1}(p_1) \Big[\frac{\gamma^\mu \not{p}_B \gamma^\nu + 2\gamma^\mu p_A^\nu}{2p_A \cdot p_B}$$

$$+ \frac{-\gamma^\nu \not{p}_2 \gamma^\mu + 2\gamma^\nu p_A^\mu}{-2p_A \cdot p_2} \Big] u^{r_A}(p_A)\ (2\pi)^4 \delta^4(p_1 + p_2 - p_A - p_B)$$

Summing and Averaging over electron spins can be done as before, using $\sum_r u^r(p)\bar{u}^r(p) = \not{p} + m$.

Summing and averaging over photon polarizations requires something analogous for $\sum_r \epsilon_\mu^r(p)^* \epsilon_\lambda^r(p)$.

For this purpose it is valid to replace:

$$\boxed{\sum_r \epsilon_\mu^r(p)^* \epsilon_\lambda^r(p) \rightarrow -g_{\mu\nu}}$$

To see that we may replace $\sum_r \epsilon^{r*}_\mu \epsilon^r_\nu$ with $-g_{\mu\nu}$, it is easiest to consider the Proca theory of the massive photon, and take the $m_\gamma \to 0$ limit.

For a massive photon we can consider the rest frame, $p^\mu = (m_\gamma, 0, 0, 0)$.
A basis of transverse polarization vectors is:

$$\epsilon^1_\mu = (0,1,0,0)$$
$$\epsilon^2_\mu = (0,0,1,0)$$
$$\epsilon^3_\mu = (0,0,0,1)$$

$$\sum_{r=1}^{3} \epsilon^{r*}_\mu \epsilon^r_\nu = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

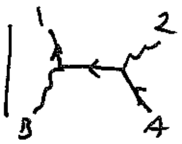
$$= -g_{\mu\nu} + \frac{p_\mu p_\nu}{m_\gamma^2}$$

The photon field couples to a conserved current, so $\epsilon^r_\mu(p)$ is contracted with $J^\mu(p)$, with $p_\mu J^\mu = 0$.

As a result of current conservation the part in $\sum_r \epsilon^{r*}_\mu \epsilon^r_\nu$ proportional to $p_\mu p_\nu$ does not contribute to the scattering amplitude, so we may replace $\sum_r \epsilon^{r*}_\mu \epsilon^r_\nu$ with $-g_{\mu\nu}$.

Since this result is independent of m_γ, we may take the $m_\gamma \to 0$ limit and reproduce the massless theory.
A better argument based on gauge invariance and Ward identities will be discussed next semester.

Note: You should average over polarizations with a $1/3$ for a massive vector field, and with a $1/2$ for a massless vector field.

$\frac{1}{4} \sum_{\substack{\text{spins} \\ \text{polarizations}}} \Big|$ 

$+$ 

$\Big|^2$

$$= \frac{e^4}{4} g_{\mu\rho} g_{\nu\sigma} \, \mathrm{Tr}\left[(\not{p}_1 + m)\left(\frac{\gamma^\mu \not{p}_B \gamma^\nu + 2\gamma^\mu p_A^\nu}{2 p_A \cdot p_B} + \frac{\gamma^\nu \not{p}_2 \gamma^\mu - 2\gamma^\nu p_A^\mu}{2 p_A \cdot p_2} \right) \right.$$

$$\left. (\not{p}_A + m)\left(\frac{\gamma^\sigma \not{p}_B \gamma^\rho + 2\gamma^\rho p_A^\sigma}{2 p_A \cdot p_B} + \frac{\gamma^\rho \not{p}_2 \gamma^\sigma - 2\gamma^\sigma p_A^\rho}{2 p_A \cdot p_2} \right) \right] \left((2\pi)^4 \delta^4 (p_1 + p_2 - p_A - p_B) \right)^2$$

where we used our previous trace identities, and

$$\begin{aligned} \overline{\gamma^\rho \not{p}_B \gamma^\sigma} &= \gamma^0 \gamma^{\sigma\dagger} \not{p}_B^\dagger \gamma^{\rho\dagger} \gamma^0 \\ &= \gamma^0 \gamma^0 \gamma^{\sigma\dagger} \gamma^0 \, \gamma^0 \not{p}_B^\dagger \gamma^0 \, \gamma^0 \gamma^{\rho\dagger} \gamma^0 \\ &= \gamma^\sigma \not{p}_B \gamma^\rho \end{aligned}$$

Using the trace identities for products of γ-matrices we could evaluate the many terms in the lowest order scattering amplitude squared.
This is tedious but straightforward, and left as an exercise.
Factoring out the $\left((2\pi)^4 \delta^4 (p_1 + p_2 - p_A - p_B)\right)^2$, which we will interpret soon, the scattering amplitude squared becomes:

$$\boxed{= 2e^4 \left[\frac{p_A \cdot p_2}{p_A \cdot p_B} + \frac{p_A \cdot p_B}{p_A \cdot p_2} + 2m^2 \left(\frac{1}{p_A \cdot p_B} - \frac{1}{p_A \cdot p_2} \right) + m^4 \left(\frac{1}{p_A \cdot p_B} - \frac{1}{p_A \cdot p_2} \right)^2 \right]}$$

Next we will turn this into a <u>scattering cross section</u>...