Wick Diagrams

Wick's theorem allows us to rewrite time ordered products of free fields in terms of normal ordered products and contractions.

We are interested in the $S$-matrix, which involves time integrals of the interaction Hamiltonian,

$$H_i(t) = H'(\phi_i, \phi_i, t), \quad \phi_i = \text{interaction picture free field}.$$ 

For free fields $\phi_i = e^{iH_0 t} \phi_0 e^{-iH_0 t} = \phi_0$, i.e., the Heisenberg picture fields, we have always been working with.

Consider $\mathcal{H}' = \int d^3x \, e^{-\bar{\Psi} \gamma^m \gamma^a \Psi A_m}$

At second order in $\mathcal{H}'$ the $S$-matrix includes

$$S = \frac{(-ie)^2}{2!} \int d^4x_1 d^4x_2 \, T \left( \bar{\Psi} \gamma^m \gamma^a A_m(x_1) \, \bar{\Psi} \gamma^a \gamma^b A_b(x_2) \right)$$

$$= \frac{(-ie)^2}{2!} \int d^4x_1 d^4x_2 \, \bar{\Psi} \gamma^m \gamma^a A_m(x_1) \, \bar{\Psi} \gamma^a \gamma^b A_b(x_2) + \text{all contractions}$$

by Wick's theorem.
We can enumerate all the terms in the Wick expansion diagrammatically.

For each factor of $\bar{\Psi}^{\mu} \gamma^\mu A_\mu$ appearing in the Wick expansion of the time ordered exponential, draw a vertex labeled by the spacetime coordinate.

**Example:**

$$\Gamma_{\mu} = -ie \int d^4 x_i : \bar{\Psi}^{\mu} \gamma^\mu A_\mu(x_i) :$$

**Notation**

- Outgoing arrow represents $\Psi(x_i)$ — creates electrons, annihilates positrons
- Ingoing arrow represents $\bar{\Psi}(x_i)$ — annihilates electrons, creates positrons
- Wavy line represents $A_\mu(x_i)$ — creates and annihilates photons

**Example:** At second order in $\mathcal{E}$ the term in the Wick expansion w/ no contractions is represented as:

$$\Gamma_{\mu} \Gamma_{\nu} = \frac{(-ie)^2}{2!} \int d^4 x_i, d^4 x_2 : \bar{\Psi}^{\mu} \gamma^\mu A_\mu(x_i) \bar{\Psi}^{\nu} \gamma^\nu A_\nu(x_2) :$$

For each contraction connect the lines correspondingly to the contracted fields.
Example:

\[ \frac{(-ie)^2}{2!} \int d^4 x_1 \, d^4 x_2 : \bar{\psi} \gamma^\mu \gamma^\nu \psi A_\mu(x_1) \bar{\psi} \gamma^\nu \psi A_\nu(x_2) : \]

For now it doesn't matter in which direction we draw the arrows, as long as at each vertex there is one ingoing (\( \psi \)) and one outgoing (\( \bar{\psi} \)).

The contraction is a C-number function of the coordinates, so in the example above there are creation and annihilation operators for electrons only, and positrons only.

Note that the arrows on fermion lines always flow in one direction:

\[ \frac{(-ie)^2}{2!} \int d^4 x_1 \, d^4 x_2 : \bar{\psi} A_\mu(x_1) \bar{\psi} A_\nu(x_2) : \]

But \( \bar{\psi} \psi = 0 \), so

\[ \frac{(-ie)^2}{2!} \int d^4 x_1 \, d^4 x_2 : \bar{\psi} A_\mu(x_1) \bar{\psi} A_\nu(x_2) : = 0 \]

Similarly, \( \bar{\psi} \psi = 0 \), so

\[ = 0 \]
Since the arrows always line up we can simplify the notation:

\[
\begin{array}{c}
\begin{array}{c}
\downarrow \downarrow \\
1 & 2
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\rightarrow \rightarrow \\
1 & 2
\end{array}
\end{array}
\]

Note that \( \begin{array}{c}
\begin{array}{c}
\downarrow \downarrow \\
1 & 2
\end{array}
\end{array} \) and \( \begin{array}{c}
\begin{array}{c}
\rightarrow \rightarrow \\
2 & 1
\end{array}
\end{array} \) count as distinct terms in the Wick expansion, although after integrating over \( \chi_1 \) and \( \chi_2 \) they are equal.

On the other hand, \( \begin{array}{c}
\begin{array}{c}
\downarrow \downarrow \\
1 & 2
\end{array}
\end{array} \) and \( \begin{array}{c}
\begin{array}{c}
\rightarrow \rightarrow \\
2 & 1
\end{array}
\end{array} \) correspond to the same term in the Wick expansion and should only be counted once.

Similarly, \( \begin{array}{c}
\begin{array}{c}
\bigcirc \bigcirc \\
1 & 2
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\bigcirc \bigcirc \\
2 & 1
\end{array}
\end{array} \) are only counted once because there is only one way to contract all of the fields at one vertex with all of the fields at the other vertex. Both diagrams represent

\[
\frac{(-ie)^2}{2!} \int d^4 \chi_1 d^4 \chi_2 \int i \chi_1 \chi_2 A(x_1) \chi_1 \chi_2 \psi_{\bar{A}}(x_2):$
\]
It is useful to classify diagrams as either connected or disconnected.

A connected diagram is drawn in one connected piece that wouldn’t fall apart if "held by one leg," e.g. \( \begin{array}{c}
\text{connected diagram}
\end{array} \) , etc.

A disconnected diagram cannot be drawn that way, e.g. \( \begin{array}{c}
\text{disconnected diagram}
\end{array} \) .

We will prove a powerful theorem:

\[
\Sigma \text{all Wick diagrams} = \exp[\Sigma \text{connected Wick diagrams}].
\]

Let \( D \) be a diagram with \( n(D) \) vertices. In the Wick expansion there is an operator associated with this diagram

\[
\Theta(D) = \frac{\Theta(D)}{n(D)!}.
\]

We have pulled out the normal ordering and the \( \frac{1}{n(D)!} \) from the expansion of the exponential in

\[
S = T \exp \left[ -i \int_{0}^{\infty} dt H(t) \right].
\]
For example, for $D = \overbrace{\cdots \cdots}^{k}$ the operator $\Theta(D)$ is

$$
\Theta(D) = (-ie)^k \int d^4 x_1 d^4 x_2 \, \overline{\Psi} \gamma^m \gamma^\mu A_m(x_1) \, \overline{\Psi} \gamma^n \gamma^\nu \Psi_{\nu}(x_2).
$$

Two diagrams are of the same pattern if they differ just by permuting the labels at the vertices $1, 2, \ldots, n(D)$.

For example, $\overbrace{\cdots \cdots}^{D_1}$ and $\overbrace{\cdots \cdots}^{D_2}$ are of the same pattern. They correspond to the operators $\Theta(D_1) = (-ie)^2 \int d^4 x_1 d^4 x_2 \, \overline{\Psi} \gamma^\mu \gamma^\nu \Psi_{\nu}(x_1) \, \overline{\Psi} \gamma^\mu \gamma^\nu \Psi_{\nu}(x_2)$

and $\Theta(D_2) = (-ie)^2 \int d^4 x_1 d^4 x_2 \, \overline{\Psi} \gamma^\mu \gamma^\nu \Psi(x_1) \, \overline{\Psi} \gamma^\mu \gamma^\nu \Psi(x_2)$.

After integration over $\overbrace{x_1, \ldots, x_{n(D)}}$ all diagrams of the same pattern are identical. However, not all permutations of the vertices give distinct diagrams, so the $n(D)!$ permutations of $1, \ldots, n(D)$ do not in general completely cancel the $\frac{1}{n(D)!}$ from the expansion of the exponential.

For example, $\overbrace{\cdots \cdots}^{D_1}$ and $\overbrace{\cdots \cdots}^{D_2}$ are not distinct. Both diagrams represent,

$$
\Theta(D) = (-ie)^2 \int d^4 x_1 d^4 x_2 \, \overline{\Psi} \gamma^m \gamma^\mu A_m(x_1) \, \overline{\Psi} \gamma^n \gamma^\nu \Psi_{\nu}(x_2).$$
For any pattern there is a symmetry factor
\[ \text{S}(\text{D}) = \# \text{ permutations which have no effect on the diagram D}. \]

Summing over all diagrams of the same pattern as D gives
\[ \frac{\Theta(\text{D})}{\text{S}(\text{D})}. \]

Let \( D_1, D_2, \ldots \) be a complete set of connected diagrams, with one diagram of each connected pattern. A general diagram has \( n_r \) components of pattern \( D_r \). For example, if \( D_1 = \Psi \psi \),

then \( \Psi_2 \psi_2 \) corresponds to \( \overline{\Psi}_2 \overline{\psi}_2(\chi_1) \overline{\Psi}_2 \overline{\psi}_2(\chi_2) \)

\[ \equiv \Theta(D_1)^2, \quad \text{i.e.} \quad n_1 = 2. \]

More generally, we can write:
\[ \Theta(\text{D}) = \prod_{r=1}^{\infty} [\Theta(D_r)]^{n_r}. \]

Summing over all diagrams w/ pattern D gives
\[ \frac{\Theta(\text{D})}{\text{S}(\text{D})}, \quad \text{where} \quad \text{S}(\text{D}) = \prod_{r=1}^{\infty} (\text{S}(D_r))^{n_r} n_r!. \]

Moreover, for each copy of \( D_r \), the symmetry factor for each exchange of the \( n_r \) copies of \( D_r \) is

\[ \frac{\Theta(\text{D})}{\text{S}(\text{D})} = \prod_{r=1}^{\infty} \frac{[\Theta(D_r)]^{n_r}}{\text{S}(D_r)^{n_r} n_r!} \]
Summing over all patterns $\mathcal{D}$ is equivalent to summing over all sets $\{n_r\}$. Hence,

$$
\sum \text{all Wick diagrams} = \sum_{n_r=0}^{\infty} \sum_{n_{r+1}=0}^{\infty} \cdots \sum_{n_{r+s}=0}^{\infty} \frac{[\Theta (\mathcal{D}_r)]^{n_r}}{s! n_r!} \prod_{r=1}^{d} 
$$

$$
= \prod_{r=1}^{d} \left( \sum_{n_r=0}^{\infty} \frac{[\Theta (\mathcal{D}_r)/s! n_r]}{n_r!} \right) 
$$

$$
= \prod_{r=1}^{d} \exp \left[ \frac{\Theta (\mathcal{D}_r)/s! n_r]}{n_r!} \right] 
$$

$$
= \exp \left[ \sum_{r=1}^{d} \frac{\Theta (\mathcal{D}_r)/s! n_r]}{n_r!} \right] 
$$

$$
= \exp \left[ \sum \text{all connected Wick diagrams} \right] 
$$

As a result of this theorem, calculation of the $S$-matrix only requires a calculation of connected diagrams.

This theorem is also useful in statistical mechanics. The free energy is the log of the partition function $T e^{-\beta H}$. In many systems, calculation of the partition function has a diagrammatic expansion, and because of this theorem, the free energy becomes a sum over connected diagrams.
Feynman Diagrams

Our Wick diagrams encode the perturbative expansion of the $S$-matrix. Matrix elements of $S$ are encoded by Feynman diagrams, which are just Wick diagrams w/ labels for incoming and outgoing states.

**Example**: Electron-Electron scattering at lowest order $\Theta(e^2)$.

We want $\langle p_1, r_1; p_2, r_2 | (S-1) | p_4, r_4; p_8, r_8 \rangle$.

The $-1$ is because we are not interested in the no-scattering process $p_4 = p_1, p_8 = p_2$ or $p_4 = p_2, p_8 = p_1$.

Recall the relativistic normalization for the states:

$|p_4, r_4; p_8, r_8 \rangle = \sqrt{2\nu_{p_4}} \sqrt{2\nu_{p_8}} |\nu_{p_4} \nu_{p_8} \uparrow \uparrow \downarrow \downarrow + 10\rangle$.

The Wick diagram contributes to $e^- e^- \rightarrow e^- e^-$ at $\Theta(e^2)$ is:

$$\frac{(-ie)^2}{2!} \int d^4 x_1 \ldots d^4 x_2 : \bar{\psi} \gamma^\mu V_A(x_1) \psi \gamma^\nu V_A(x_2) :$$
The matrix element we are after is:

\[ \frac{\alpha^2}{2i} \int d\gamma_1 d\gamma_2 \langle P_2, \eta_2, P_1, \eta_1 | \overline{\psi}\gamma^\mu \psi(x_1) \overline{\psi}\gamma^\nu \psi(x_2) | P_2, \eta_2, P_1, \eta_1 \rangle \]

\[ \overline{A}_\mu(x_1) A_\nu(x_2) \]

Consider the underlined part:

\[ \sqrt{2w_1} \sqrt{2w_2} \sqrt{2w_a} \sqrt{2w_b} \langle 0 | q_{\eta_2}^c q_{\eta_1}^a | \overline{\psi}\gamma^\mu \psi(x_1) \overline{\psi}\gamma^\nu \psi(x_2) | q_{\eta_2}^{c*} q_{\eta_1}^{a*} + 0 \rangle \]

\[ \overline{\psi}(x_i) \] contains \( q_{x_i}^{c*} \) and \( b_{x_i}^{c*} \)

\[ \psi(x_i) \] contains \( q_{x_i}^c \) and \( b_{x_i}^c \)

The only nonvanishing contribution is from the \( q_{x_i}^c q_{x_i}^{c*} q_{x_i}^{a*} q_{x_i}^a \) term in the normal-ordered product.

From \( \overline{\psi}(x_i) \), \( \psi(x_i) \), \( \overline{\psi}(x_i) \), \( \psi(x_i) \)

Written this out:

\[ \int \frac{d^4 x_1 d^4 x_2 d^4 k_1' d^4 k_2'}{(2\pi)^6 \sqrt{2w_1} \sqrt{2w_2} \sqrt{2w_a} \sqrt{2w_b}} \langle 0 | q_{\eta_2}^c q_{\eta_1}^a \int d^4 x_3 d^4 x_4 d^4 k_3' d^4 k_4' \frac{1}{(2\pi)^6 \sqrt{2w_1} \sqrt{2w_2} \sqrt{2w_a} \sqrt{2w_b}} \sum_{s_i, s_i, s_i'} \frac{q_{x_i}^{s_i} q_{x_i}^{s_i*} | 0 \rangle \langle 0 | q_{x_i}^{s_i'} q_{x_i}^{s_i'*} q_{x_i}^{a*} q_{x_i}^a | q_{\eta_2}^{c*} q_{\eta_1}^{a*} + 0 \rangle}{\sqrt{\sum_{s_i, s_i, s_i'}}} \times \frac{u^{s_i'}(k_1') u^{s_i}(k_1) u^{s_i'}(k_2') u^{s_i}(k_2)}{u^{s_i'}(k_1) u^{s_i}(k_1) u^{s_i'}(k_2) u^{s_i}(k_2)} \times \exp i [k_1 \cdot x_1 - k_1' \cdot x_1 + k_2 \cdot x_2 - k_2' \cdot x_2] \]

From inserting a complete set of states here.
Anticommuting the $q$'s and $q^\dagger$'s using
$$\{q^s, q^p \} = (2\pi)^3 \delta^3(\vec{r} - \vec{r}') \delta_{rs}$$
gives four terms:

$$\langle 0 | q_{a_2}^s q_{p_1}^s q_{k_1}^s q_{k_2}^s | 0 \rangle \langle 0 | q_{k_1}^o q_{k_2}^o q_{p_4}^o q_{p_8}^o | 0 \rangle$$

$$= (2\pi)^{12} \left[ \delta^3(k_1 - p_1) \delta^3(k_2 - p_2) \delta^3(k_4' - p_4) \delta^3(k_8' - p_8) \\
\times \delta_{s,s'} \delta_{s',r_2} \delta_{s',r_4} \delta_{s',r_8} \\
- \delta^3(k_1 - p_1) \delta^3(k_2 - p_2) \delta^3(k_4' - p_4) \delta^3(k_8' - p_8) \\
\times \delta_{s,s'} \delta_{s',r_4} \delta_{s',r_2} \delta_{s',r_8} \right]$$

$$+ \left( (k_1, k_2) \leftrightarrow (k_3, k_1), \ (k_1', k_2') \leftrightarrow (k_3', k_1') \right)$$

Exchanging $(k_1, k_2) \leftrightarrow (k_3, k_1)$ and $(k_1', k_2') \leftrightarrow (k_3', k_1')$ is equivalent to exchanging $\pi$, and $\pi_2$ in the integrals, and cancels the $\frac{i}{2\pi}$ left over from the Wick expansion.

In QED all diagrams w/ external lines (scattering states) will have the $\frac{i}{2\pi}$ from the expansion of the exponential cancelled.

Doing the $k_1, k_2, k_1', k_2'$ integrals, the $\sim$ factors cancel.

Putting back the contraction and the $k$, $\pi$ integrals gives the matrix element we started with, which now takes the form:
\[
\left\{ \begin{array}{l}
\frac{(-ie)^2}{2!} \int d^4 x_1 \, d^4 x_2 \, \overline{A}_\mu(x_1) A_\nu(x_2) \\
\left[ (-\bar{u} \gamma^\mu u) \bar{\sigma}^{(P_1)} u \bar{\gamma}_\nu u \bar{\tau}^{(P_2)} \right] \right.
\left. e^{-i P_1 \cdot x_1} e^{-i P_2 \cdot x_2} e^{-i P_3 \cdot x_2} e^{-i P_4 \cdot x_2} \right.
\left. + \bar{u} \gamma^\mu u \sigma^{(P_2)} \bar{\gamma}_\nu u \bar{\tau}^{(P_1)} \right]
\frac{e^{-i \frac{g_{\mu\nu}}{k^2+i\epsilon}}}{e^{-i K \cdot (x_1 - x_2)}} \left[ \ast \right]
\end{array} \right.
\]

Doing the \( x_1, x_2 \) integrals gives \&-functions:

\[
\left. 2 \times \frac{(-ie)^2}{2!} \int d^4 K \, \frac{i \, g_{\mu\nu}}{k^2+i\epsilon} \left[ \bar{u} \gamma^\mu u \bar{\sigma}^{(P_1)} u \bar{\gamma}_\nu u \bar{\tau}^{(P_2)} \right] \right.
\left. \times \delta^4(K+P_1-P_3) \delta^4(K-P_2+P_4) \right] (2\pi)^8 \left[ 1 \right]
\left. - \bar{u} \gamma^\mu u \bar{\sigma}^{(P_2)} \bar{\gamma}_\nu u \bar{\tau}^{(P_1)} \right]
\left. \times \delta^4(K+P_2-P_3) \delta^4(K-P_1+P_4) \right] (2\pi)^8 \left[ 2 \right]
\left. = -e^2 \left[ \frac{i \, g_{\mu\nu}}{(P_1-P_3)^2+i\epsilon} \bar{u} \gamma^\mu u \bar{\sigma}^{(P_1)} u \bar{\gamma}_\nu u \bar{\tau}^{(P_2)} \right] \right.
\left. \times \delta^4(K+P_1-P_3) \delta^4(K-P_2+P_4) \right] \left[ 1 \right]
\left. - \frac{i \, g_{\mu\nu}}{(P_2-P_3)^2+i\epsilon} \bar{u} \gamma^\mu u \bar{\tau}^{(P_2)} \bar{\gamma}_\nu u \bar{\sigma}^{(P_1)} \right]
\left. \times \delta^4(K+P_2-P_3) \delta^4(K-P_1+P_4) \right] \left[ 2 \right]
\left. = -e^2 \left[ 1 + 2 \right] \right. 
\]
There is a diagram that goes along with each of these terms:

\[ e^2 [(D + \alpha)] \quad \Rightarrow \quad \text{Diagram} \quad + \quad \text{Diagram} \quad \uparrow \quad \text{Time} \]

\[ P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8 \]

- The delta functions enforce 4-momentum conservation at each vertex: \( K = P_A - P_1 = P_2 - P_8 \) in second diagram,
  \( K = P_4 - P_2 = P_1 - P_8 \) in first diagram.

There is an overall 4-momentum conserving 5-function left over \( (2\pi)^4 \delta^4 (P_1 + P_2 - P_A - P_8) \).

We won't always draw arrows outside the diagrams to show the direction of momentum. Our convention, unless stated otherwise, will be states in the past have momentum flowing into the vertex.
State in the future have momentum flowing out of the vertex.

- For each ingoing electron we get a \( u^c(p) \).
- For each outgoing electron we get a \( \overline{u}^c(p) \).
- For the internal line we get a \( \frac{-i\gamma_{\mu}}{k^2 + i\epsilon} \).

Next we will generalize this calculation and develop the Feynman rules for QED.