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Wick Diagrams

Wick's theorem allows us to rewrite time ordered products of free fields in terms of normal ordered products and contractions.

We are interested in the S-matrix, which involves time integrals of the interaction Hamiltonian, $H_I(t) = H'(\phi_I, \partial_\mu \phi_I, t)$, ϕ_I = interaction picture free field.

For free fields $\phi_I = e^{iH_0^s t} \phi_S e^{-iH_0^s t} = \phi_H$, ie, the Heisenberg picture fields we have always been working with.

Consider QED : $H' = \int d^3x \; e \bar{\Psi} \gamma^\mu \Psi A_\mu$

At second order in e the S-matrix includes

$$S \supset \frac{(-ie)^2}{2!} \int d^4x_1 \, d^4x_2 \; T\left(\bar{\Psi}\gamma^\mu \Psi A_\mu(x_1) \; \bar{\Psi}\gamma^\nu \Psi A_\nu(x_2) \right)$$

$$= \frac{(-ie)^2}{2!} \int d^4x_1 \, d^4x_2 \; :\bar{\Psi}\gamma^\mu \Psi A_\mu(x_1) \; \bar{\Psi}\gamma^\nu \Psi A_\nu(x_2):$$

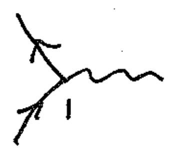
$$+ \text{ all contractions}$$

by Wick's theorem.

We can enumerate all the terms in the Wick expansion diagramatically.

For each factor of $\bar{\Psi}\gamma^\mu\Psi A_\mu$ appearing in the Wick expansion of the time ordered exponential draw a <u>vertex</u> labeled by the spacetime coordinate.

<u>Example:</u>



$$= -ie\int d^4x_1 \; :\bar{\Psi}\gamma^\mu\Psi A_\mu(x_1):$$

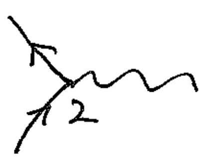
<u>Notation</u>

Outgoing arrow represents $\bar{\Psi}(x_1)$ — creates electrons, annihilates positrons

Ingoing arrow represents $\Psi(x_1)$ — annihilates electrons, creates positrons

Wavy line represents $A_\mu(x_1)$ — creates and annihilates photons

<u>Example:</u> At second order in e the term in the Wick expansion w/ no contractions is represented as:



$$= \frac{(-ie)^2}{2!}\int d^4x_1\, d^4x_2 \; :\bar{\Psi}\cancel{A}\Psi(x_1)\, \bar{\Psi}\cancel{A}\Psi(x_2):$$

For each contraction connect the lines corresponding to the contracted fields.

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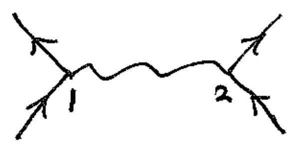
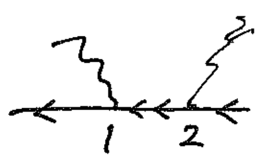
Example:  $= \frac{(-ie)^2}{2!}\int d^4x_1\, d^4x_2 \; :\bar\psi\gamma^\mu\psi \overbrace{A_\mu(x_1)\bar\psi\gamma^\nu\psi A_\nu(x_2)}:$

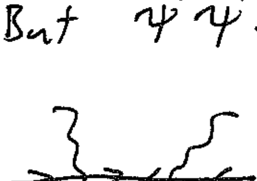
For now it doesn't matter in which direction we draw the arrows, as long as at each vertex there is one ingoing (ψ) and one outgoing ($\bar\psi$).

The contraction is a C-# function of the coordinates, so in the example above there are creation and annihilation operators for electrons ~~resp.~~ and positrons only.

Note that the arrows on fermion lines always flow in one direction:

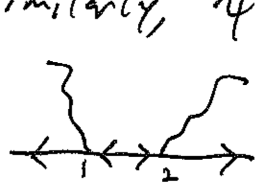
 $= \frac{(-ie)^2}{2!}\int d^4x_1\, d^4x_2 \; :\bar\psi \not{A}\overbrace{\psi(x_1)\;\bar\psi}\not{A}\psi(x_2):$

But $\overbrace{\bar\psi\,\bar\psi}=0$, so

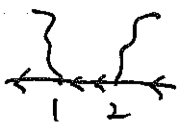
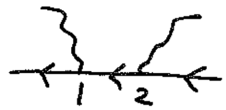
 $= \frac{(-ie)^2}{2!}\int d^4x_1\, d^4x_2 \; :\overbrace{\bar\psi \not{A}\psi(x_1)\;\bar\psi}\not{A}\psi(x_2):$

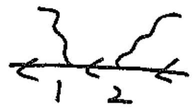
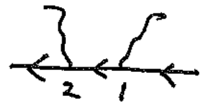
$= 0$

Similarly, $\overbrace{\psi\psi}=0$, so

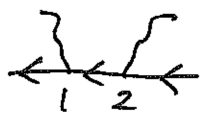
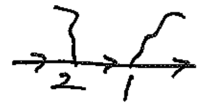
 $= 0.$

Since the arrows always line up we can simplify the notation:

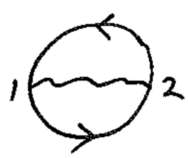
 = 

Note that  and  count

as distinct terms in the Wick expansion, although after integrating over x_1 and x_2 they are equal.

On the other hand,  and 

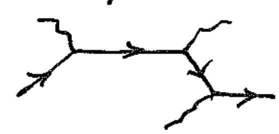
correspond to the same term in the Wick expansion and should only be counted once.

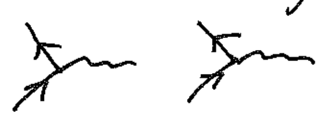
Similarly,  =  are only

counted once because there is only one way to contract all of the fields at one vertex with all of the fields at the other vertex.
Both diagrams represent

$$\frac{(-ie)^2}{2!}\int d^4x_1\, d^4x_2 \; : \bar{\psi}\gamma^\mu\psi A_\mu(x_1)\; \bar{\psi}\gamma^\nu\psi A_\nu(x_2) :$$

It is useful to classify diagrams as either <u>connected</u> or <u>disconnected</u>.

A connected diagram is drawn in one connected piece that would not fall apart if "held by one leg," eg. , , etc.

A disconnected diagram cannot be drawn that way, eg. 

We will prove a powerful theorem:

$$\boxed{\Sigma \text{ all Wick diagrams} = :\exp\left[\Sigma \text{ connected Wick diagrams}\right]:}$$

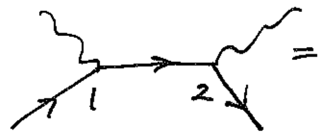
Let D be a diagram with $n(D)$ vertices. In the Wick expansion there is an operator associated with this diagram $\dfrac{:O(D):}{n(D)!}$

We have pulled out the normal ordering and the $1/n(D)!$ from the expansion of the exponential in $S = T\exp\left[-i\int_{-\infty}^{\infty} dt\, H_I(t)\right]$.

For example, for $D =$  the operator $\mathcal{O}(D)$ is

$$\mathcal{O}(D) = (-ie)^2 \int d^4x_1\, d^4x_2 \;\; \bar{\psi}\gamma^\mu\psi A_\mu(x_1) \;\; \bar{\psi}\gamma^\nu\psi A_\nu(x_2).$$

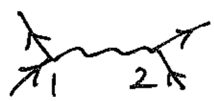
Two diagrams are of the same <u>pattern</u> if they differ just by permuting the labels at the vertices $1, 2, \ldots, n(D)$.

For example,  $= D_1$ and  $= D_2$

are of the same pattern. They <u>correspond to the</u> operators $\mathcal{O}(D_1) = (-ie)^2 \int d^4x_1\, d^4x_2 \;\; \bar{\psi}\not{A}\psi(x_1) \;\; \bar{\psi}\not{A}\psi(x_2)$

and $\mathcal{O}(D_2) = (-ie)^2 \int d^4x_1\, d^4x_2 \;\; \bar{\psi}\not{A}\psi(x_1) \;\; \bar{\psi}\not{A}\psi(x_2)$.

After integration over $x_1, \ldots, x_{n(D)}$ all diagrams of the same pattern are identical. However, not all permutations of the vertices give distinct diagrams, so the $n(D)!$ permutations of $1, \ldots, n(D)$ do not in general completely cancel the $1/n(D)!$ from the expansion of the exponential.

For example,  and  are not

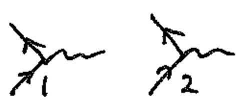
distinct. Both diagrams represent,

$$\mathcal{O}(D) = (-ie)^2 \int d^4x_1\, d^4x_2 \;\; \bar{\psi}\gamma^\mu\psi A_\mu(x_1) \;\; \bar{\psi}\gamma^\nu\psi A_\nu(x_2)$$

For any pattern there is a symmetry factor $S(D)$ = # permutations which have no effect on the diagram D.

Summing over all diagrams of the same pattern as D gives $\dfrac{:\Theta(D):}{S(D)}$.

Let $D_1, D_2, \ldots$ be a complete set of connected diagrams, with one diagram of each connected pattern. A general diagram has n_r components of pattern D_r. For example, if $D_1 =$ 

then  corresponds to $:\bar{\Psi} A\!\!\!/ \Psi(x_1)\ \bar{\Psi} A\!\!\!/ \Psi(x_2): \equiv \,:\Theta(D_1)^2:$, i.e. $n_1 = 2$.

More generally, we can write:
$$:\Theta(D): \;=\; :\prod_{r=1}^{\infty} [\Theta(D_r)]^{n_r}:$$

Summing over all diagrams w/ pattern D gives $\dfrac{:\Theta(D):}{S(D)}$, where $S(D) = \prod_{r=1}^{\infty} [S(D_r)]^{n_r}\, n_r!$

symmetry factor for each copy of D_r

Exchange of the n_r copies of D_r.

So, $$\frac{:\Theta(D):}{S(D)} = \prod_{r=1}^{\infty} \frac{:[\Theta(D_r)]^{n_r}:}{S(D_r)^{n_r}\, n_r!}$$

Summing over all patterns D is equivalent to summing over all sets $\{n_r\}$. Hence,

$\boxed{\sum \text{all Wick diagrams}}$

$$= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots : \prod_{r=1}^{\infty} \frac{[\theta(D_r)]^{n_r}}{S(D_r)^{n_r}\, n_r!} :$$

$$= : \prod_{r=1}^{\infty} \left(\sum_{n_r=0}^{\infty} \frac{[\theta(D_r)/S(D_r)]^{n_r}}{n_r!} \right) :$$

$$= : \prod_{r=1}^{\infty} \exp\left[\theta(D_r)/S(D_r)\right] :$$

$$= : \exp\left[\sum_{r=1}^{\infty} \theta(D_r)/S(D_r) \right] :$$

$$\boxed{= \ : \exp\left[\sum \text{all connected Wick diagrams}\right] :}$$

As a result of this theorem calculation of the S-matrix only requires a calculation of connected diagrams.

This theorem is also useful in statistical mechanics. The free energy is the log of the partition function $\mathrm{Tr}\, e^{-\beta H}$. In many systems calculation of the partition function has a diagrammatic expansion, and because of this theorem the free energy becomes a sum over connected diagrams.

Feynman Diagrams

Our Wick diagrams encode the perturbative expansion of the S-matrix. Matrix elements of S are encoded by Feynman diagrams, which are just Wick diagrams w/ labels for ingoing and outgoing states.

Example: Electron-Electron scattering at lowest order $\mathcal{O}(e^2)$.

We want $\langle p_1, r_1 ; p_2, r_2 | (S-1) | p_A, r_A ; p_B, r_B \rangle$

Outgoing electron momenta, spins

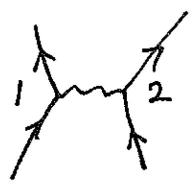
← Ingoing electron momenta, spins

The -1 is because we are not interested in the no-scattering process $p_A = p_1, p_B = p_2$ or $p_A = p_2, p_B = p_1$.

Recall the relativistic normalization for the states:

$$|p_A, r_A ; p_B, r_B \rangle = \sqrt{2\omega_{p_A}}\sqrt{2\omega_{p_B}}\, a_{p_A}^{r_A\dagger} a_{p_B}^{r_B\dagger} |0\rangle.$$

The Wick diagram contributing to $e^- + e^- \to e^- + e^-$ at $\mathcal{O}(e^2)$ is:



$$\frac{(-ie)^2}{2!}\int d^4x_1\, d^4x_2 \, : \bar\psi \gamma^\mu \psi A_\mu(x_1)\, \bar\psi\gamma^\nu\psi A_\nu(x_2) :$$

The matrix element we are after is,

$$\frac{(ie)^2}{2!}\int d^4x_1\, d^4x_2\, \underbrace{\langle p_1, r_1; p_2, r_2|:\bar\Psi\gamma^\mu\Psi(x_1)\,\bar\Psi\gamma^\nu\Psi(x_2):|p_A, r_A; p_B, r_B\rangle}$$

$$\times \overbrace{A_\mu(x_1)A_\nu}(x_2)\rangle$$

Consider the underlined part:

$$\sqrt{2\omega_{p_1}}\sqrt{2\omega_{p_2}}\sqrt{2\omega_{p_A}}\sqrt{2\omega_{p_B}}\,\langle 0|a^{r_2}_{p_2}a^{r_1}_{p_1}:\bar\Psi\gamma^\mu\Psi(x_1)\,\bar\Psi\gamma^\nu\Psi(x_2):a^{r_A+}_{p_A}a^{r_B+}_{p_B}|0\rangle$$

$\bar\Psi(x_i)$ contains $a^{r_i+}_{k_i}$ and $b^{r_i}_{k_i}$

$\Psi(x_i)$ contains $a^{r_i}_{k_i}$ and $b^{r_i+}_{k_i}$

The only nonvanishing contribution is from the $a^{r_1+}_{k_1}a^{r_2+}_{k_2}a^{r_1'}_{k_1'}a^{r_2'}_{k_2'}$ term in the normal-ordered product.

From $\bar\Psi(x_1)$ $\quad\bar\Psi(x_2)\quad\Psi(x_1)\quad\Psi(x_2)$

Writing this out:

From normal ordering

$$-\sqrt{2\omega_{p_1}}\sqrt{2\omega_{p_2}}\sqrt{2\omega_{p_A}}\sqrt{2\omega_{p_B}}\,\langle 0|a^{r_2}_{p_2}a^{r_1}_{p_1}\int\frac{d^3k_1\,d^3k_1'\,d^3k_2\,d^3k_2'}{(2\pi)^{12}\sqrt{2\omega_{k_1}2\omega_{k_1'}2\omega_{k_2}2\omega_{k_2'}}}$$

$$\sum_{\substack{s_1,s_1'\\ s_2,s_2'}} a^{s_1+}_{k_1}a^{s_2+}_{k_2}|0\rangle\langle 0|a^{s_1'}_{k_1'}a^{s_2'}_{k_2'}a^{r_A+}_{p_A}a^{r_B+}_{p_B}|0\rangle$$

$$\times\,\bar u^{s_1}(k_1)\gamma^\mu u^{s_1'}(k_1')\,\bar u^{s_2}(k_2)\gamma^\nu u^{s_2'}(k_2')$$

$$\times\,\exp i\left[k_1\cdot x_1 - k_1'\cdot x_1 + k_2\cdot x_2 - k_2'\cdot x_2\right]$$

From inserting a complete set of states here.

Anticommuting the a's and $a^\dagger$'s using
$\{a_K^s, a_p^{r\dagger}\} = (2\pi)^3 \delta^3(\vec{k}-\vec{p})\delta^{rs}$
gives four terms:

$$\langle 0| a_{p_2}^{r_2} a_{p_1}^{r_1} a_{k_1}^{s_1\dagger} a_{k_2}^{s_2\dagger}|0\rangle \langle 0| a_{k_1'}^{s_1'} a_{k_2'}^{s_2'} a_{p_A}^{r_A\dagger} a_{p_B}^{r_B\dagger}|0\rangle$$

$$= (2\pi)^{12}\Big[\delta^3(k_1-p_1)\delta^3(k_2-p_2)\delta^3(k_2'-p_A)\delta^3(k_1'-p_B) \times \delta^{r_1 s_1}\delta^{r_2 s_2}\delta^{s_2' r_A}\delta^{s_1' r_B}$$
$$- \delta^3(k_1-p_1)\delta^3(k_2-p_2)\delta^3(k_1'-p_A)\delta^3(k_2'-p_B) \times \delta^{r_1 s_1}\delta^{r_2 s_2}\delta^{s_1' r_A}\delta^{s_2' r_B}\Big]$$
$$+ \Big((k_1,k_2)\longleftrightarrow(k_2,k_1)\,,\ (k_1',k_2')\leftrightarrow(k_2',k_1')\Big)$$

Exchanging $(k_1,k_2)\Leftrightarrow(k_2,k_1)$ and $(k_1',k_2')\Leftrightarrow(k_2',k_1')$ is equivalent to exchanging x_1 and x_2 in the integrals, and cancels the $\frac{1}{2!}$ left over from the Wick expansion.

In QED all diagrams w/ <u>external lines</u> (scattering states) will have the $\frac{1}{n!}$ from the expansion of the exponential cancelled

Doing the k_1, k_2, k_1', k_2' integrals, the $\sqrt{2\omega}$ factors cancel.

Putting back the contraction and the x_1, x_2 integrals gives the matrix element we started with, which now takes the form:

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$$\frac{(-ie)^2}{2!}\int d^4x_1\, d^4x_2\; \overbrace{A_\mu(x_1)A_\nu(x_2)}$$

$$[*]\;\left\{\begin{aligned}&\Big[\Big(\bar{u}^{r_1}(p_1)\gamma^\mu u^{r_B}(p_B)\,\bar{u}^{r_2}(p_2)\gamma^\nu u^{r_A}(p_A)\, e^{ip_1\cdot x_1} e^{-ip_B\cdot x_1} e^{ip_2\cdot x_2} e^{-ip_A\cdot x_2}\\ &+\bar{u}^{r_2}(p_2)\gamma^\mu u^{r_B}(p_B)\,\bar{u}^{r_1}(p_1)\gamma^\nu u^{r_A}(p_A)\, e^{ip_2\cdot x_1} e^{-ip_B\cdot x_1} e^{ip_1\cdot x_2} e^{-ip_A\cdot x_2}\Big)\\ &\qquad + (x_1 \leftrightarrow x_2)\Big]\end{aligned}\right.$$

$$= \frac{(-ie)^2}{2!}\int d^4x_1\, d^4x_2\, \frac{d^4k}{(2\pi)^4}\,\frac{-ig_{\mu\nu}}{k^2+i\epsilon}\, e^{ik\cdot(x_1-x_2)}\;[*]$$

Doing the x_1, x_2 integrals gives δ-functions:

From $(x_1 \leftrightarrow x_2)$

$$= 2\times\frac{(-ie)^2}{2!}\int\frac{d^4k}{(2\pi)^4}\,\frac{ig_{\mu\nu}}{k^2+i\epsilon}\Big[\bar{u}^{r_1}(p_1)\gamma^\mu u^{r_B}(p_B)\,\bar{u}^{r_2}(p_2)\gamma^\nu u^{r_A}(p_A)$$

$$\times\,\delta^4(k+p_1-p_B)\,\delta^4(k-p_2+p_A)\,(2\pi)^8$$

$$-\bar{u}^{r_2}(p_2)\gamma^\mu u^{r_B}(p_B)\,\bar{u}^{r_1}(p_1)\gamma^\nu u^{r_A}(p_A)$$

$$\times\,\delta^4(k+p_2-p_B)\,\delta^4(k-p_1+p_A)\,(2\pi)^8\Big]$$

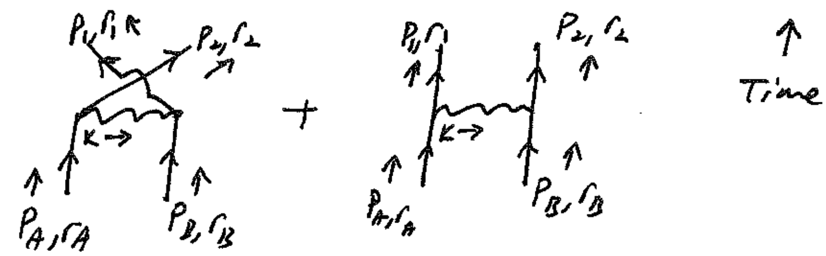
$$= -e^2\Big[\frac{ig_{\mu\nu}}{(p_1-p_B)^2+i\epsilon}\,\bar{u}^{r_1}(p_1)\gamma^\mu u^{r_B}(p_B)\,\bar{u}^{r_2}(p_2)\gamma^\nu u^{r_A}(p_A)(2\pi)^4\delta^4(p_1+p_2-p_A-p_B)$$

$$-\frac{ig_{\mu\nu}}{(p_2-p_B)^2+i\epsilon}\,\bar{u}^{r_2}(p_2)\gamma^\mu u^{r_B}(p_B)\,\bar{u}^{r_1}(p_1)\gamma^\nu u^{r_A}(p_A)(2\pi)^4\delta^4(p_1+p_2-p_A-p_B)\Big]$$

$$\equiv -e^2\big[①+②\big]$$

There is a diagram that goes along with each of these terms:

$-e^2[① + ②] \rightsquigarrow$



- The delta-functions enforce 4-momentum conservation at each vertex: $K = p_A - p_1 = p_2 - p_B$ in second diagram
  $K = p_A - p_2 = p_1 - p_B$ in first diagram

There is an overall 4-momentum conserving δ-function left over $(2\pi)^4 \delta^4(p_1 + p_2 - p_A - p_B)$.

We won't always draw arrows outside the diagrams to show the direction of momentum. Our convention, unless stated otherwise, will be states in the past have momentum flowing into the vertex.
States in the future have momentum flowing out of the vertex.

- For each ingoing electron we got a $u^r(p)$.
- For each outgoing electron we got a $\bar{u}^r(p)$.

- For the internal line we got a $\dfrac{-i g_{\mu\nu}}{k^2 + i\epsilon}$

Next we will generalize this calculation and develop the Feynman rules for QED.