Quantization of the Dirac Spiner Field - Part 2

We are now ready to attempt to canonically quantize the theory of the free Dirac spinor field.

The field satisfies the Dirac equation, and is decomposed by

$$\Psi(x) = \int\frac{d^3k}{(2\pi)^{3/2}\omega_k} \left( e^{-i\omega_k t} \sum_{\alpha=1,2} \epsilon^{\alpha} \psi_k \left( \frac{\sigma}{\tau} \right) + e^{i\omega_k t} \sum_{\alpha=1,2} \epsilon^{\alpha} \sigma_k \psi_k \left( \frac{\sigma}{\tau} \right) \right)$$

The canonical momentum conjugate to $\Psi$ is

$$\Pi_\Psi = \frac{\partial}{\partial \partial_\Psi} = i \Psi^+$$

To determine the commutation relations for the operators $\sigma_k$, $\sigma_k^+$, $\psi_k$, and $\sigma_k^+$, we impose the ETCR's on $\Psi$ and $\Pi_\Psi$:

$$[\Psi_\alpha(x, t), \Psi_\beta(x', t)] = 0$$

$$[\Pi_\Psi_\alpha(x, t), \Pi_\Psi_\beta(x', t)] = 0$$

$$[\Psi_\alpha(x, t), \Pi_\Psi_\beta(x', t)] = i \delta^3(\vec{x} - \vec{y}) \delta_{\alpha\beta}$$

where $\alpha$ and $\beta = 1, 2, 3, 4$ are Dirac spinor indices.
\[ + \frac{d^3 k \cdot d^3 k'}{(2\pi)^6 2\omega_k 2\omega_k'} \left( e^{-i(k \cdot x - k' \cdot x')} \sum_{\sigma} \left[ q_k^\sigma q_{k'}^{s+} \right] u_{\sigma}^{s+}(k) v_{s}(k')^* \right. \\
\left. + e^{-i(k \cdot x + k' \cdot x')} \sum_{\sigma} \left[ b_k^\sigma b_{k'}^{s+} \right] v_{\sigma}(k) u_{s}(k')^* \right. \\
\left. + e^{-i(k \cdot x - k' \cdot x')} \sum_{\sigma} \left[ q_k^s b_{k'}^{s+} \right] u_{\sigma}^{s+}(k) v_{s}(k')^* \right. \\
\left. + e^{-i(k \cdot x + k' \cdot x')} \sum_{\sigma} \left[ b_k^s q_{k'}^{s+} \right] v_{\sigma}(k) u_{s}(k')^* \right). \]

If \[ [q_k^s, b_{k'}^{s+}] = [b_k^s, q_{k'}^{s+}] = 0 \]
\[ [q_k^r, q_{k'}^{s+}] = [b_k^r, b_{k'}^{s+}] = \frac{\hbar}{2\pi} \delta^3(r - r') \delta^3(r', r). \]

Then \[ [\psi(x, t), \psi^+(y, t)] = \frac{i}{\sqrt{2\omega_k}} \left( \sum_{\sigma} u_{\sigma}^{s+}(k) \right) \]
\[ = i \left( \frac{d^3 k}{(2\pi)^3 2\omega_k} \right \} - \frac{d^3 k}{(2\pi)^3 2\omega_k} \left( (k \cdot m) \gamma^0 e^{i \cdot \vec{k} \cdot \vec{x} - \frac{\hbar}{2} \cdot \vec{p}) \right. + \left. (k-m) \gamma^0 e^{-i \cdot \vec{k} \cdot \vec{x} + \frac{\hbar}{2} \cdot \vec{p}) \right) \]
\[ = i \left( \frac{d^3 k}{(2\pi)^3 2\omega_k} \right \} - \frac{d^3 k}{(2\pi)^3 2\omega_k} \left( (w_k \gamma^0 + k \cdot \gamma^i m) + (w_k \gamma^0 - k \cdot \gamma^i m) \right) \gamma^0 e^{i \cdot \vec{k} \cdot \vec{x} - \frac{\hbar}{2} \cdot \vec{p}) \right. + \left. (w_k \gamma^0 - k \cdot \gamma^i m) \gamma^0 e^{-i \cdot \vec{k} \cdot \vec{x} + \frac{\hbar}{2} \cdot \vec{p}) \right) \]
\[ = i \left( \frac{d^3 k}{(2\pi)^3 2\omega_k} \right \} - \frac{d^3 k}{(2\pi)^3 2\omega_k} \left( \frac{\hbar}{2} \cdot \vec{p} \right) \gamma^0 e^{i k \cdot \vec{x} - \frac{\hbar}{2} \cdot \vec{p}) \right. + \left. e^{-i k \cdot \vec{x} + \frac{\hbar}{2} \cdot \vec{p}) \right) \]
\[ = i \left( \frac{d^3 k}{(2\pi)^3 2\omega_k} \right \} - \frac{d^3 k}{(2\pi)^3 2\omega_k} \left( \frac{\hbar}{2} \cdot \vec{p} \right) \gamma^0 e^{i k \cdot \vec{x} - \frac{\hbar}{2} \cdot \vec{p}) \right. + \left. e^{-i k \cdot \vec{x} + \frac{\hbar}{2} \cdot \vec{p}) \right) \]
\[ = i j^3(x, t, \tau) 1, \ as \ desired. \]
Similarly, if \([q^x_k, q^y_k] = [b^x_k, b^y_k] = [\bar{q}^x_k, \bar{b}^x_k] = 0\)

then \([^\psi(x), \psi(y)]_x \xrightarrow{x \to y} 0 \)

And if \([q^x_k, q^y_{k'}] = [b^x_k, b^y_{k'}] = [\bar{q}^x_k, \bar{b}^x_{k'}] = 0\)

then \([^\psi(x), \psi(y)]_x \xrightarrow{x \to y} 0 \)

Next, we calculate the Hamiltonian in terms of the creation and annihilation operators.

**Dirac Field Hamiltonian:** 

\[ H = \mp \int d^4x \, \overline{\psi} (-i \gamma^0 \partial_0 + m) \psi \]

\[ = \mp \int d^4x \, \overline{\psi} (i \gamma^0 \partial_0) \psi \]

by the Dirac equation.

\[ H = \mp \int d^3x \int \frac{d^3k d^3k'}{(2\pi)^6 2\omega_k 2\omega_{k'}} \left(e^{i k_\nu (x_\nu - x'_\nu)} (-i \omega_{k'}) \sum_{\alpha \beta} q^{\alpha+} b^{\beta+} \overline{\psi}(k) i \gamma_\alpha \gamma_\beta u^s(k') \right. \]

\[ + e^{-i k_\nu (x_\nu - x'_\nu)} (i \omega_k) \sum_{\alpha \beta} b^{\alpha+} q^{\beta+} \overline{\psi}(k') i \gamma_\alpha \gamma_\beta u^s(k) \]

\[ + e^{i k_\nu (x_\nu - x'_\nu)} (i \omega_{k'}) \sum_{\alpha \beta} q^{\alpha+} b^{\beta+} \overline{\psi}(k) i \gamma_\alpha \gamma_\beta u^s(k') \]

\[ + e^{-i k_\nu (x_\nu - x'_\nu)} (-i \omega_{k}) \sum_{\alpha \beta} b^{\alpha+} q^{\beta+} \overline{\psi}(k') i \gamma_\alpha \gamma_\beta u^s(k) \left. \right) \]

The \( \int d^3x \) integrals give delta-functions as usual. The delta functions allow us to do the \( \int d^3k' \) integrals.
Then, \( H = \sqrt{\frac{\delta^4 \langle k \rangle}{2 \omega^2 \omega_k}} \sum_{\text{ns}} \left( q^r + q^s \right) \tilde{u}^r(\vec{k}) \tilde{r}^s(\vec{k}) \)

\[ - \frac{\delta^4 \langle k \rangle}{\omega^2} \tilde{v}^r(\vec{k}) \delta^0 \tilde{v}^s(\vec{k}) \]

\[ - \frac{\delta^4 \langle k \rangle}{\omega^2} \tilde{u}^r(\vec{k}) \delta^0 \tilde{u}^s(\vec{k}) \left( \frac{2i \omega^0 \omega_k}{\omega^2} \right) \]

\[ + \frac{\delta^4 \langle k \rangle}{\omega^2} \tilde{v}^r(\vec{k}) \delta^0 \tilde{v}^s(\vec{k}) \left( \frac{2i \omega^0 \omega_k}{\omega^2} \right) \]

To calculate expressions like \( \tilde{u}^r(\vec{k}) \delta^0 \tilde{u}^s(\vec{k}) \) we can use the fact that \( \tilde{u}^r \delta^0 \tilde{u}^s \) is the time component of a 4-vector \( \tilde{u} \delta^0 \tilde{u}^* \), and boost the expression from the rest frame.

In the rest frame, the Dirac equation gives

\[ \delta^0 u = u \quad \text{(rest frame)} \]
\[ \delta^0 v = -v \]

By the orthogonality relations,

\[ \tilde{u}^r \delta^0 u^s = \tilde{u}^r u^s = 2m \delta^{rs} \]
\[ \tilde{v}^r \delta^0 v^s = -\tilde{v}^r v^s = 2m \delta^{rs} \]
\[ \tilde{u}^r \delta^0 v^s = \tilde{v}^r \delta^0 u^s = 0 \]

Boosting to a general frame,

\[ \tilde{u}^r(\vec{k}) \delta^0 u^s(\vec{k}) = 2\omega_k \delta^{rs} \]
\[ \tilde{v}^r(\vec{k}) \delta^0 v^s(\vec{k}) = 2\omega_k \delta^{rs} \]
\[ \tilde{u}^r(\vec{k}) \delta^0 v^s(\vec{k}) = 0 \]
\[ \tilde{v}^r(\vec{k}) \delta^0 u^s(\vec{k}) = 0 \]
Exercise: Using the solutions to the Dirac equation, it is also the case that:

\[
\overline{\psi}(\vec{r}) \gamma^0 \psi(-\vec{r}) = \overline{\psi}(\vec{r}) \gamma^0 \psi(-\vec{r}) = 0
\]

Using these relations, the Hamiltonian becomes:

\[
H = \pm \int \frac{d^3k}{(2\pi)^3} \, 2\omega_k \sum \frac{1}{r} (q_k^r + q_k^{-r} - b_k^r + b_k^{-r})
\]

\[
= \pm \int \frac{d^3k}{(2\pi)^3} \, \omega_k \sum \frac{1}{r} (q_k^r + q_k^{-r} - b_k^r + b_k^{-r})
\]

If we define the vacuum such that \( q_k^r |0\rangle = b_k^{-r} |0\rangle = 0 \), then:

The energy of the state \( b_{k_1}^r \cdots b_{k_n}^r |0\rangle \) is \( \mp \sum_{i=1}^{n} \omega_{k_i} \).

The energy of the state \( q_{k_1}^r \cdots q_{k_n}^r |0\rangle \) is \( \pm \sum_{i=1}^{n} \omega_{k_i} \).

The particles and antiparticles carry opposite sign energy, so the Hamiltonian is unbounded below.

Neither choice of sign in the Lagrangian, nor a redefinition of the vacuum by say, \( b_{k}^{-r} |0\rangle \neq 0 \), will cure this problem.

\( \Rightarrow \) Equal-time commutation relations for Dirac spinor fields leads to disaster.
The solution to the problem lies in the statistics of the particles created by \( q^{+} \) and \( b^{+} \).
We have inadvertently tried to make a theory of bosonic electrons and failed.

What makes these particles bosons? The Hilbert space of states is constructed as in the scalar field, and we can choose the same normalizations, i.e.

\[
| \Psi, r \rangle = \sqrt{2w_1} q^{+}_{E_2} + 10 \rangle, \text{ etc.}
\]

Consider the two-electron state,

\[
| k_{1r} j k_{2s} \rangle = \sqrt{2w_1 \sqrt{2w_2}} q^{+}_{E_2} q^{+}_{E_1} + 10 \rangle
\]

Switching the labels \( k_1 \leftrightarrow k_2 \) and \( r \leftrightarrow s \), we have

\[
| k_{2s} j k_{1r} \rangle = \sqrt{2w_1 \sqrt{2w_2}} q^{+}_{E_2} q^{+}_{E_1} + 10 \rangle
\]

\[
= | k_{1r} j k_{2s} \rangle \quad \text{because } [q^{+}_{E_1}, q^{+}_{E_2}] = 0.
\]

Hence, these multiparticle states are symmetric under exchange, i.e. bosons.

If we instead had had \([q^{+}_{E_1}, q^{+}_{E_2}] = 0\), then we would have obtained

\[
| k_{1r} j k_{2s} \rangle = -| k_{2s} j k_{1r} \rangle,
\]

as appropriate for fermions.
For the Dirac spinor field, we must assume equal-time anticommutation relations to obtain a description of fermionic spin-\(\frac{1}{2}\) particles with positive energy:

\[
\{\psi(x), \psi^+(\mathbf{x}')\} = i \frac{\gamma^2(x - x')}{\mathbf{x} \cdot \mathbf{x}'} \delta(x - x')
\]

\[
\{\psi(x), \psi(\mathbf{x})\} = 0
\]

\[
\{\psi(x)^+, \psi(\mathbf{x})^+\} = 0
\]

Then we would deduce

\[
\{q^+_E, q^+_E\} = \{b^+_E, b^+_E\} = (2\pi)^3 \int \delta^3(\mathbf{E} - \mathbf{E'}) \delta^{(5)}
\]

All other anticommutators vanish.

Now consider the Hamiltonian. We still have

\[
H = \pm \int \frac{d^3k}{(2\pi)^3} \omega_k \sum_E (q^+_E q^+_E - b^+_E b^+_E)
\]

Suppose we redefine \(b^+_E \leftrightarrow b^+_E\) while maintaining the conditions defining the vacuum \(q^+_E|0> = b^+_E|0> = 0\) (for the newly redefined \(b^+_E\)).

In other words, we write the field as \(\psi(x)\)

\[
\psi(x) = \int \frac{d^3k}{(2\pi)^3 \sqrt{2\omega_k}} \left( e^{-i\mathbf{k} \cdot \mathbf{x}} \sum_{\mathbf{r}} q_{\mathbf{r}} \psi(x) + e^{i\mathbf{k} \cdot \mathbf{x}} \sum_{\mathbf{r}} b^+_\mathbf{r} \psi(x) \right)
\]
The Hamiltonian is now

\[
H = \int \frac{d^3k}{(2\pi)^3} \, \sum \left( q^r + q^s - b^r b^s + \frac{1}{2} \right)\left( q^r_k + q^s_k + b^r_k b^s_k \right) - \int \frac{d^3k}{(2\pi)^3} \, \sum \frac{\omega_k \cdot 2 \delta^2(0)}{2} \frac{1}{\omega_k}
\]

There's an infinite constant, but we're used to that by now. We can redefine the Hamiltonian by removing the constant term, or we can (equivalently) normal order.

**Normal Ordering for Fermions**

In order to take into account the minus signs from anti-commuting fermionic creation or annihilation operators, we define normal ordering for fermions as follows:

\[
\begin{align*}
\langle q^r + q^s \rangle &= q^r_k + q^s_k, \\
\langle q^s K^r \rangle &= -q^r_k, q^s_k \\
\end{align*}
\]

and similarly exchanging either or both \( q \)'s with \( b \)'s.

More generally normal ordering for fermions moves all \( q^r \)'s and \( b^r \)'s to the left of \( q^s \)'s and \( b^s \)'s, with a factor of \((-1)\) for each permutation of operators required to move them to the appropriate order.
The normal-ordered Hamiltonian is

\[
\hat{H}_n = \sum_{\mathbf{k}} \frac{\hbar^2 k^2}{2m} \omega_k \ \left( q^+_k q_k + b^+_k b_k \right)
\]

From now on we'll just call this \( \hat{H} \).

Note that we get commutatorshere because \( \hat{H} \) is quadratic in fermionic operators.

\[
[H, q_k^+] = \omega_k q_k^+, \quad [H, q_k^-] = -\omega_k q_k^-
\]

\[
[H, b_k^+] = \omega_k b_k^+, \quad [H, b_k^-] = -\omega_k b_k^-
\]

With the vacuum defined by \( q_k^+ \vert 0 \rangle = b_k^+ \vert 0 \rangle = 0 \), the energy of the state \( q_{k_1}^+ \ldots q_{k_n}^+ b_{n+1}^+ \ldots b_{k_{\text{max}}}^+ \vert 0 \rangle \)

\[
H q_{k_1}^+ \ldots b_{k_{\text{max}}}^+ \vert 0 \rangle = \left( \sum_{k=1}^{\text{max}} \omega_k \right) q_{k_1}^+ \ldots b_{k_{\text{max}}}^+ \vert 0 \rangle.
\]

One-particle states are normalized as before:

\[
\vert E, q_e^- \rangle = \sqrt{2\omega_k} \ q_e^+ \vert 0 \rangle \quad \text{\( e^- \) created by \( q_e^+ \)}
\]

\[
\vert E, r_e^+ \rangle = \sqrt{2\omega_k} \ b_e^+ \vert 0 \rangle \quad \text{\( e^+ \) created by \( b_e^+ \)}
\]

\[
\langle E', r_e^- \vert E, s_e^- \rangle = \frac{2\omega_k}{\hbar} (k+1)^{1/2} \langle E', r_e^+ \vert E, s_e^+ \rangle = \frac{2\omega_k}{\hbar} (k+1)^{1/2} \langle E', r_e^- \vert E, s_e^- \rangle = 0
\]
Comment on Spin vs. Statistics

We have learned that a bosonic theory of spin-$\frac{1}{2}$ fields is inconsistent. We would have been led to similar inconsistencies if we had tried to quantize the scalar field as a fermion, i.e. w/ anticommutation relations. In relativistic QFT (in 4 dimensions) there is a deep connection between spin and statistics: integer spin fields must be bosons, and $\frac{1}{2}$-integer spin fields must be fermions.