What is Quantum Field Theory?

A field is a physical quantity associated to each point in spacetime.

Quantum Field Theory is concerned with quantization of dynamical systems of fields.

This course will focus on relativistic quantum systems. At energies \( E \geq mc^2 \) relativity is important, and particles can be produced. For example

\[ p + p \rightarrow p + p + \pi^0 \]
\[ p + p \rightarrow p + p + p + \overline{p} \]

Hence, we are forced to consider systems of many particles.

Even at lower energies, many-particle processes contribute to observables. Consider perturbative theory in nonrelativistic quantum mechanics (NRQM).

\[ H \rightarrow H + SV \quad \delta E_0 = \langle 0 | SV | 1 \rangle + \sum_n \frac{| \langle 0 | SV | n \rangle |^2}{E_0 - E_n} + \ldots \]

High-precision effects include relativistic corrections of order \( (\frac{v}{c})^2 \). Intermediate states with extra particles contribute corrections of order

\[ \frac{E}{mc^2} \approx \frac{mv^2}{mc^2} = (\frac{v}{c})^2 \quad \text{— same as relativistic corrections, so can't be ignored.} \]
Recall the motivation for the Schrödinger Eqn.

1923: Louis de Broglie — suggests material particles can act like waves.

\[ \psi(x, t) = A \exp\left[ i \frac{\mathbf{k} \cdot \mathbf{x} - \omega t}{\hbar} \right] \text{ for free particle} \]

\[ \begin{align*}
\mathbf{k} &= \frac{\mathbf{p}}{\hbar} \\
E &= \hbar \omega \end{align*} \]

Einstein-de Broglie relations.

(1925-1926: Heisenberg, Born, Jordan, Pauli — Matrix Quantum Mechanics)

1926: Schrödinger Eqn.

\[ E = \frac{\mathbf{p}^2}{2m} \text{ together w/ Einstein-de Broglie relations} \]

Consistent w/ plane wave solutions if.

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi \]

To include interactions postulate the general relationship \( E \to i\hbar \frac{\partial}{\partial t} \) consistent w/ discussion \( \mathbf{p} \to -i\hbar \nabla \) above.

\[ E = \frac{\mathbf{p}^2}{2m} + V(x) \rightarrow i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V(x) \psi \] — Nonrelativistic Schrödinger Eqn.
Following the logic of the non-relativistic Schrödinger Eqn, Schrödinger, Klein, Gordon wrote down the Relativistic Schrödinger Eqn = Klein-Gordon Eqn

\[ E^2 = p^2 c^2 + m^2 c^4 \rightarrow \frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial t^2} = -\frac{\hbar^2 c^2}{2m} \nabla^2 \psi + m^2 c^4 \psi \]

Plane-Wave Solutions: \( \psi(x, t) = A \exp \left[ i \left( \frac{E \cdot x - wt}{\hbar} \right) \right] + B \exp \left[ -i \left( \frac{E \cdot x - wt}{\hbar} \right) \right] \)

where \[ \frac{\hbar^2}{2m} \omega^2 = \frac{\hbar^2 c^2}{2m} E^2 + m^2 c^4 \]

Dispersion relation

\(-\) relate frequency and wavevector.

Q: Can the relativistic wave function \( \psi \) have a probabilistic interpretation as in non-relativistic QM?

A: No.

**Non-relativistic:**

\[ i \hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V(x) \psi \quad (1) \]

\[ -i \hbar \frac{\partial \psi^*}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi^* + V(x) \psi^* \quad (2) \]

Multiply (1) by \( \psi^* \), (2) by \( \psi \), and subtract:

\[ i \hbar \left( \psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi^*}{\partial t} \right) = \frac{-\hbar^2}{2m} \left( \psi^* \nabla^2 \psi - \psi \nabla^2 \psi^* \right) \]

\[ i \hbar \frac{\partial}{\partial t} \| \psi \|^2 = -\frac{\hbar^2}{2m} \nabla \cdot \left( \psi^* \nabla \psi - \psi \nabla \psi^* \right) \]

\[ \frac{\partial}{\partial t} \| \psi \|^2 + \frac{\hbar}{m} \nabla \cdot \text{Im}(\psi^* \nabla \psi) = 0 \]

- Resembles current conservation \( \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0 \)

\[ \rho = \| \psi \|^2 = \text{probability density} \]

\[ \vec{J} = \frac{\hbar}{m} \text{Im}(\psi^* \nabla \psi) = \text{probability current} \]
Integrate current conservation eqn over \( \mathbb{V} \):

\[
\int_{\mathbb{V}} d^3x \frac{\partial \rho}{\partial t} = -\int_{\mathbb{V}} d^3x \nabla \cdot \mathbf{J}
\]

\[
= -\int_{\mathbb{V}} d^3x \nabla \cdot \mathbf{J} = 0 \quad \text{as } \mathbb{V} \to \mathbb{R}^3 \quad \text{(all space)}
\]

\[
\frac{d}{dt} \int_{\mathbb{V}} d^3x \rho(\mathbb{x}, t) = \int_{\mathbb{V}} d^3x 1 |\Psi|^2 = 0
\]

1. Hence, the total probability is conserved.
   If \( \int_{\mathbb{V}} d^3x 1 |\Psi(\mathbb{x}, t)|^2 = 1 \) at some time \( t \), then the same is true at all times.

2. Also note that \( |\Psi|^2 \geq 0 \).

Conditions 1 and 2 are the reason \( |\Psi|^2 \) may be interpreted as a probability density.

Revisit for the relativistic wavefunction:

\[
\frac{-\hbar^2 \partial^2 \Psi}{\partial t^2} = -\frac{\hbar^2 c^2}{2m_e} \nabla^2 \Psi + m_e c^2 \Psi \quad (1')
\]

\[
\frac{-\hbar^2 \partial^2 \Psi^*}{\partial t^2} = -\frac{\hbar^2 c^2}{2m_e} \nabla^2 \Psi^* + m_e c^2 \Psi^* \quad (2')
\]

Multiply (1') by \( i \Psi^* \), (2') by \( i \Psi \) and subtract.

\[
i \frac{\partial}{\partial t} \left( \frac{\partial \Psi}{\partial t} - \frac{\partial \Psi^*}{\partial t} \right) = i c^2 \nabla \cdot (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*)
\]

This is of the form \( \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \), with

\[
\rho = \frac{i \hbar}{2mc^2} (\Psi^* \frac{\partial \Psi}{\partial t} - \Psi \frac{\partial \Psi^*}{\partial t}) = -\frac{\hbar}{2mc^2} \text{Im} (\Psi^* \frac{\partial \Psi}{\partial t})
\]

\[
\mathbf{J} = -\frac{i \hbar}{2mc^2} (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) = \frac{\hbar}{mc^2} \text{Im} (\Psi^* \nabla \Psi)
\]
\[ \frac{d}{dt} \int d^3x \rho = \frac{d}{dt} \int d^3x \text{Im}(\psi \frac{\partial \psi}{\partial t}) = 0 \]

The quantity \( \int d^3x \text{Im}(\psi \frac{\partial \psi}{\partial t}) \) is conserved, but the density \( \rho = \text{Im}(\psi \frac{\partial \psi}{\partial t}) \) is not positive semidefinite.

\[ \rightarrow \] The analog of condition (2) for the non-relativistic probability density is not satisfied.

Hence, \( \rho \) is not a probability density in this case.

Other difficulties of the Klein-Gordon eqn:

- Including electromagnetic coupling, predictions for fine structure of hydrogen spectrum disagreed with detailed measurements by Paschen.

- There are negative-energy solutions \( E = \pm \sqrt{\frac{\hbar^2}{2m} c^2 + m^2 c^4} \)

  \[ \rightarrow \] There is no ground state. What would prevent a state from decaying to successively lower energy states?
The Dirac Equation (1928)

Dirac reasoned that the essential difference between the relativistic and non-relativistic Schrödinger eqns is that the former is 2nd order in time derivatives, while the latter is 1st order.

Q: Can an eqn of the form \( i \hbar \frac{\partial \psi}{\partial t} = H \psi \)
be consistent with Lorentz invariance?

A: Yes, if there are enough \( \psi \)'s and \( H \) is a matrix of differential operators.

Lorentz invariance relates spatial and time coordinates, so conjecture that \( H \) is linear in spatial derivatives.

\[
i \hbar \frac{\partial \psi}{\partial t} = -i \hbar c \left( \alpha^1 \frac{\partial \psi}{\partial x^1} + \alpha^2 \frac{\partial \psi}{\partial x^2} + \alpha^3 \frac{\partial \psi}{\partial x^3} \right) \psi + \beta mc^2 \psi
\]

Dirac Equation

If \( \alpha^i \) were just \( c \)-numbers (complex numbers) then the Dirac eqn would not transform consistently under rotations of the coordinates.

What if \( \alpha^i, \beta \) are \( N \times N \) matrices, and \( \psi \) has \( N \) components?

\[
i \hbar \frac{\partial \psi}{\partial t} = -i \hbar c \sum_{k=1}^{N} \left( \alpha^1 \frac{\partial \psi_k}{\partial x^1} + \alpha^2 \frac{\partial \psi_k}{\partial x^2} + \alpha^3 \frac{\partial \psi_k}{\partial x^3} \right) \psi_k + \sum_{k=1}^{N} \beta_i \psi_k mc^2 \psi_k
\]

\[= \sum_{k=1}^{N} H \psi_k \psi_k \]
Iterate the Dirac eqn,

\[-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = -\frac{\hbar^2}{2} \sum_{k=1}^{3} \frac{1}{2} \alpha^k \beta^k + \alpha^k \beta \cdot \beta \, \frac{\partial^2 \psi}{\partial x^k \partial x^k} \]

\[+ \frac{\hbar c m c^2}{i} \sum_{k=1}^{3} \left( \alpha^k \beta^k + \beta \alpha^k \right) \frac{\partial \psi}{\partial x^k} \]

\[+ \beta^2 m^2 c^4 \psi \]

To recover \( E^2 = \beta c^2 + m^2 c^4 \) for plane wave solutions, we arrange that each component of \( \psi \) satisfy the Klein-Gordon eqn,

\[-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = -\frac{\hbar^2}{2} \partial^2 \psi + m^2 c^4 \psi \]

In that case,

1. \[\alpha^k \beta^k + \alpha^k \beta \cdot \beta \frac{\partial}{\partial x^k} = 2 \delta^{k\ell} \frac{1}{2} \]

   \[\text{Kronecker } \delta^{k\ell} = \begin{cases} 1 & \text{if } k = \ell \\ 0 & \text{if } k \neq \ell \end{cases} \]

2. \[\alpha^k \beta + \beta \alpha^k = 0 \]

3. \[\beta^2 = 1_{\text{WZW}} \]

- It follows from (1) and (3) that \((\alpha^k)^2 = \beta^2 = 1_{\text{WZW}}\)

\[\rightarrow \text{eigenvalues of } \alpha^k, \beta \text{ are } \pm 1 \]

- \(0 = (\alpha^k \beta + \beta \alpha^k) \beta = \alpha^k + \beta \alpha^k \beta \)

\[\text{Tr } \alpha^k = -\text{Tr } \beta \alpha^k \beta = -\text{Tr } \beta^2 \alpha^k = -\text{Tr } \alpha^k \]

\[\rightarrow \text{Tr } \alpha^k = 0 \quad \text{cyclicity of trace} \quad \beta^2 = 1_{\text{WZW}} \]

- Similarly, \(0 = \alpha^k (\alpha^k \beta + \beta \alpha^k) = \beta + \alpha^k \beta \alpha^k \)

\[\rightarrow \text{Tr } \beta = 0 \]
\[ \text{Tr} \alpha^k = \Sigma (\text{eigenvalues of } \alpha^k) = 0 \]
\[ \implies \# + 1, -1 \text{ eigenvalues equal. Similarly for } \beta \]
\[ \implies \alpha, \beta \text{ must be even-dimensional matrices.} \]

Hermitean \Rightarrow \alpha, \beta \text{ Hermitean matrices.}

In summary, we need 4 mutually anticommuting, even-dimensional Hermitean matrices.

2x2 Hermitean matrices: spanned by 3 mutually anticommuting Pauli matrices \( \sigma^k \) and the unit matrix \( \mathbb{1} \).
\[ \implies 2 \times 2 \text{ 13 out.} \]

(However, note that if the mass \( m = 0 \), then there is no matrix \( \beta \) in the Dirac eqns, and we would only need 3 matrices - the Pauli \( \sigma \)-matrices would work.)

What about 4x4 matrices?

\[ \text{Dirac basis: } \alpha^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

Each entry is a 2x2 matrix.

\[ \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]
What about the probability density?

\[ i\hbar \frac{\partial \psi}{\partial t} = -i\hbar c \sum_{k=1}^{3} \alpha \frac{\partial \psi}{\partial x_k} + \beta mc^2 \psi \quad (1') \]

\[-i\hbar \frac{\partial \psi^+}{\partial t} = i\hbar c \sum_{k=1}^{3} \frac{\partial \psi^+}{\partial x_k} \alpha^* \psi + mc^2 \psi^+ \quad (2') \]

Left-multiply (1') by \( \psi^+ \), right-multiply (2') by \( \psi \)

Subtract:

\[ i\hbar \left( \psi^+ \frac{\partial \psi}{\partial t} + \frac{\partial \psi^+}{\partial t} \psi \right) = -i\hbar c \sum_{k=1}^{3} \left( \frac{\psi^+ \alpha \frac{\partial \psi}{\partial x_k} + \psi \frac{\partial \psi^+}{\partial x_k} \alpha^* \psi} \right) \]

\[ \frac{\partial}{\partial t} (\psi^+ \psi) + c \sum_{k=1}^{3} \frac{\partial}{\partial x_k} (\psi^+ \alpha^* \psi) = 0 \]

This is of the form \( \frac{\partial \rho}{\partial t} + \text{div } j = 0 \) with

\[ \rho = \psi^+ \psi \]

\[ j^k = c \psi^+ \alpha^* \psi, \quad k = 1, 2, 3 \]

Note that \( \rho \geq 0 \) and is conserved, so it is a candidate for a probability density.