Physics 721, Fall 2006 Josh Erlich

Problem Set 3: Quantized Scalar Fields

Due Tuesday, October 10.

1. *Harmonic oscillators from canonical quantization*

Consider a free real scalar field $\phi(x)$ with Lagrangian,

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{m^2}{2}\phi^2.$$

Decompose $\phi(x)$ as,

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_{\mathbf{k}}}}\left(a_{\mathbf{k}}\, e^{-ik\cdot x} + a^\dagger_{\mathbf{k}}\, e^{ik\cdot x}\right),$$

where $k^0 = \omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$.

a) Solve for $a_{\mathbf{k}}$ and $a^\dagger_{\mathbf{k}}$ in terms of $\phi(x)$ and its conjugate momentum $\Pi(x)$.

b) Use the canonical commutation relations for $\phi(x)$ and $\Pi(x)$ to directly calculate the commutators $[a_{\mathbf{k}}, a_{\mathbf{k}'}]$, $[a^\dagger_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}]$ and $[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}]$.

2. *4-momentum as generator of space-time translations*

Consider a free real scalar field $\phi(x)$.

a) Express the normal-ordered Hamiltonian H and spatial momentum $\mathbf{P}$ as integrals involving raising and lowering operators, and using the harmonic oscillator commutation relations show that:

$$e^{iHt}\, a_{\mathbf{k}}\, e^{-iHt} = a_{\mathbf{k}} e^{-i\omega_{\mathbf{k}}t}$$
$$e^{iHt}\, a^\dagger_{\mathbf{k}}\, e^{-iHt} = a^\dagger_{\mathbf{k}}\, e^{i\omega_{\mathbf{k}}t}$$
$$e^{-i\mathbf{P}\cdot\mathbf{x}}\, a_{\mathbf{k}}\, e^{i\mathbf{P}\cdot\mathbf{x}} = a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$$
$$e^{-i\mathbf{P}\cdot\mathbf{x}}\, a^\dagger_{\mathbf{k}}\, e^{i\mathbf{P}\cdot\mathbf{x}} = a^\dagger_{\mathbf{k}}\, e^{-i\mathbf{k}\cdot\mathbf{x}}.$$

b) Using this, show that,

$$\phi(x) = e^{i(Ht - \mathbf{P}\cdot\mathbf{x})}\, \phi(0)\, e^{-i(Ht - \mathbf{P}\cdot\mathbf{x})}$$

Hence, the 4-momentum operator $(H, \mathbf{P})$ generates translations of the field $\phi(x)$ in space-time. This is a specific example of a general phenomenon:

the conserved charge due to a symmetry generates the corresponding symmetry transformation on the fields.

3. *Global SO(2) symmetry of a pair of real scalars*

Suppose $\phi_1(x)$ and $\phi_2(x)$ are a pair of real scalar fields with Lagrangian,

$$\mathcal{L} = \frac{1}{2} \sum_{j=1}^{2} \left((\partial_\mu \phi_j)^2 - m^2 \, \phi_j^2 \right) .$$

Each of the two fields has its own set of creation and annihilation operators. Decompose the fields in plane waves as,

$$\phi_j(x) = \int \frac{d^3k}{(2\pi)^3 \sqrt{2\omega_{\mathbf{k}}}} \left(a_{\mathbf{k}}^{(j)} \, e^{-ik\cdot x} + a_{\mathbf{k}}^{(j)\,\dagger} \, e^{ik\cdot x} \right) ,$$

where $k^0 = \omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$.

Consider the symmetry transformation,

$$\begin{aligned} \phi_1 &\rightarrow \cos\theta \, \phi_1 + \sin\theta \, \phi_2 \\ \phi_2 &\rightarrow \cos\theta \, \phi_2 - \sin\theta \, \phi_1 , \end{aligned}$$

for $0 \leq \theta < 2\pi$.

Calculate the normal-ordered charge associated with this symmetry in terms of a single d^3k integral involving the creation and annihilation operators.