Physics 721, Fall 2006 Josh Erlich

Problem Set 2: Symmetries and Conservation Laws

Due Tuesday, September 26.

1. *More practice with gamma matrices*

Show that:

$$\gamma^5 \gamma^\mu \gamma^5 = -\gamma^\mu$$

$$\mathrm{Tr}\, \not{a} = 0$$

$$\mathrm{Tr}\, \not{a} \not{b} = 4a \cdot b$$

Similarly, compute Tr $\not{a} \not{b} \not{c}$, Tr $\not{a} \not{b} \not{c} \not{d}$, Tr $\not{a} \gamma^5$, Tr $\not{a} \not{b} \gamma^5$, Tr $\not{a} \not{b} \not{c} \gamma^5$, and Tr $\not{a} \not{b} \not{c} \not{d} \gamma^5$.

The last of these will involve the constant antisymmetric tensor $\epsilon^{\mu\nu\rho\sigma}$. Use only the anticommutation relations of the 4×4 gamma matrices. Do not use an explicit representation of the matrices. Recall that $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

2. *Rotations and angular momentum conservation*

Consider a system of N point particles described by the N vectors $\mathbf{r}^a$, $a = 1, \ldots, N$, with Lagrangian,

$$L = \sum_{a=1}^{N} \frac{m_a}{2} \left|\dot{\mathbf{r}}^a\right|^2 - \sum_{a,b=1}^{N} V_{ab} \left(\left|\mathbf{r}^a - \mathbf{r}^b\right|\right).$$

Consider an infinitessimal rotation by angle θ about the axis $\mathbf{e}$,

$$\mathbf{r}^a \to \mathbf{r}^a + \theta\, \mathbf{e} \times \mathbf{r}^a.$$

a) Show that the action is invariant under rotations.

b) Show that the usual angular momentum is conserved as a consequence of rotation invariance in this system.

3. *The Dirac field and electric charge*

The Dirac Lagrangian is,

$$\mathcal{L} = \overline{\psi} \left(i \not{\partial} - m\right) \psi.$$

a) Treating ψ and $\overline{\psi}$ as independent fields, derive the equations of motion for ψ and $\overline{\psi}$. Show that the equations you derive are self consistent.

b) The transformation $\psi \to e^{i\theta}\,\psi$, $\overline{\psi} \to e^{-i\theta}\,\overline{\psi}$ is a symmetry of the theory. What is the associated 4-vector current? What is the charge? (When we quantize the theory we will identify this with the electric charge.)