Physics 721, Fall 2006 Josh Erlich
Problem Set 1: The Dirac Equation
Due Tuesday, September 19.

1. *Lorentz transformations*

If $\Lambda^\mu{}_\nu$ describes a Lorentz transformation, such that

$$x^\mu \to \Lambda^\mu{}_\nu \, x^\nu ,$$

then how do the following transform under the Lorentz transformation:

a) The Minkowski tensor, $\eta_{\mu\nu}$?

b) $\partial_\mu \phi(x) \, \partial^\mu \phi(x)$, where $\phi(x)$ is a scalar field?

c) $\frac{\partial}{\partial x^\mu} \left[\overline{\psi}(x) \, \gamma^\mu \, \psi(x) \right]$, where $\psi(x)$ is a Dirac spinor field?

d) $\overline{\psi}(x) \gamma^\mu \gamma^\nu \psi(x)$?

Prove your results by explicit computation, using the properties of $\Lambda^\mu{}_\nu$ and the specified representations of the Lorentz group.

2. *Dirac spinor representation*

The generators of rotations in three dimensions, T^a, $a = 1, 2, 3$ satisfy the SO(3) algebra, $\left[T^a, T^b \right] = i \, \epsilon^{abc} \, T^c$, where ϵ^{abc} is completely antisymmetric in a, b and c, with $\epsilon^{123} = 1$.

The analogous relations for the six generators of Lorentz transformations $J^{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3$ with $J^{\mu\nu} = -J^{\nu\mu}$, are,

$$\left[J^{\mu\nu}, J^{\rho\sigma} \right] = i \left(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho} \right) .$$

These commutation relations define the Lorentz algebra, SO(3, 1). Using the properties of the Dirac γ-matrices, show that the generators of Lorentz transformations in the Dirac spinor representation,

$$S^{\mu\nu} = \frac{i}{4} \left[\gamma^\mu, \gamma^\nu \right] ,$$

satisfy the commutation relations describing the Lorentz algebra.

3. *Chirality*

Any Dirac spinor can be decomposed into a left-handed and a right-handed part by using the **chirality projection operators**,

$$P_L = \frac{1-\gamma^5}{2}, \qquad P_R = \frac{1+\gamma^5}{2}.$$

Using the properties of γ^5 show that:

a) $P_L^2 = P_L, \quad P_R^2 = P_R, \quad P_L P_R = P_R P_L = 0.$

b) Given a Dirac spinor ψ define its left-handed part as $\psi_L \equiv P_L\,\psi$ and its right-handed part as $\psi_R \equiv P_R\,\psi$. Show that under a Lorentz transformation of ψ, the left and right-handed components of ψ transform independently. This implies that the Dirac spinor forms a **reducible representation** of the Lorentz group.

c) By acting on the Dirac equation $(i\partial\!\!\!/ - m)\,\psi = 0$ with P_L and with P_R, rewrite the Dirac equation in terms of a coupled set of equations for ψ_L and ψ_R.

d) Show that the equations for ψ_L and ψ_R decouple when $m \to 0$.

4. *Solution to free Dirac equation with momentum in arbitrary direction*

The solution to the Dirac equation for an electron moving in the $\mathbf{x}^3$-direction with momentum p^3 is, in the Weyl basis,

$$\psi(x) = u(p)\, e^{ip^\mu x_\mu},$$

where,

$$u(p) = \begin{pmatrix} \left[\sqrt{E+p^3}\left(\frac{1-\sigma^3}{2}\right) + \sqrt{E-p^3}\left(\frac{1+\sigma^3}{2}\right)\right]\,\xi \\ \left[\sqrt{E+p^3}\left(\frac{1+\sigma^3}{2}\right) + \sqrt{E-p^3}\left(\frac{1-\sigma^3}{2}\right)\right]\,\xi \end{pmatrix},$$

and ξ is a 2-component Pauli spinor.

Show that the above solution can be written as,

$$u(p) = \begin{pmatrix} \sqrt{p^\mu\sigma_\mu}\;\xi \\ \sqrt{p^\mu\overline{\sigma}_\mu}\;\xi \end{pmatrix},$$

where $\sigma^\mu = (1, \vec{\sigma})$, $\overline{\sigma}^\mu = (1, -\vec{\sigma})$. In this form the solution is valid when the momentum is in an arbitrary direction.

5. *Current conservation for the Dirac equation*

Show that,

$$\begin{aligned}\rho &= \psi^\dagger \psi \\ \vec{J} &= c\,\psi^\dagger\,\vec{\alpha}\,\psi,\end{aligned}$$

satisfy the current conservation equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0,$$

where ψ satisfies the Dirac equation for an electron coupled to a background electromagnetic field, and $\vec{\alpha}$ are the 4×4 matrices appearing in the Dirac equation.