Physics 721, Fall 2006 Josh Erlich

Midterm Exam. Due 9:30am, Tuesday, November 14.

You may consult any notes you have taken, any notes I have given you, and Peskin & Schroeder's textbook. That is all you may consult, out of fairness to your classmates. You should work alone. You are expected to spend not more than 8 hours working on the exam.

If you have any questions, feel free to email me at erlich@physics.wm.edu, or phone me on my cell phone at (757)272-2697. I will post answers to any questions I receive on the course website at:
http://physics.wm.edu/∼erlich/721F06

There is one problem on the exam, but it is divided into 12 parts. **You should do all of parts (a)-(g), and any two of parts (h)-(l).**

Massive electrodynamics

The most general Lagrangian density for the vector field including terms quadratic in the field with at most two derivatives is (up to the addition of total derivatives):

$$\mathcal{L} = -\frac{1}{2}\,\partial_\mu A_\nu\,\partial^\mu A^\nu + b\,\partial_\mu A_\nu \partial^\nu A^\mu + c\,A_\mu A^\mu.$$

(a) (10 pts) What are the Euler-Lagrange equations for this theory?

(b) (10 pts) Assume a plane wave solution of the form $A^\mu(x) = \varepsilon^\mu(\mathbf{k})e^{-ik\cdot x}$. What are the Euler-Lagrange equations in terms of ε^μ and k^μ?

(A^μ is real, but as usual we are describing two solutions at once – the real and imaginary parts of the plane wave. We can do this because the Euler-Lagrange equations are linear in this theory.)

(c) (10 pts) **Longitudinal mode:** Assume $\varepsilon^\mu(\mathbf{k}) \propto k^\mu$. What are the Euler-Lagrange equations for this *ansatz* in terms of ε^μ and k^μ? Define $k_\mu k^\mu \equiv m_L^2$. What is the longitudinal mass m_L in terms of the parameters b and c in the Lagrangian?

(d) (10 pts) **Transverse modes:** Repeat part (c) assuming that $\varepsilon_\mu k^\mu = 0$. This time define $k_\mu k^\mu \equiv m_T^2$. What is m_T?

(e) (10 pts) The longitudinal mode will not propagate if $m_L \to \infty$. What choice of b accomplishes this? Make that choice and rewrite $\mathcal{L}$ in terms of A^μ, m_T, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

This is the Proca Lagrangian of massive electrodynamics. You should recover Maxwell's theory if you take $m_T \to 0$.

The rest of the problem concerns the Proca Lagrangian you have just derived.

(f) (10 pts) If $m_T \neq 0$ then the action is not gauge invariant. In QED we had the freedom to impose a gauge condition on A_μ. Show that the Lorenz gauge condition $\partial_\mu A^\mu = 0$ follows from the equations of motion as long as $m_T \neq 0$.

(g) (10 pts) What are the equal time commutation relations? (Only impose equal time commutation relations on the complete set of initial value data. In other words, the momentum conjugate to A_0 vanishes, but you do not need to impose commutation relations involving A_0.)

Choose any two of parts (h)-(l).

(h) (15 pts) What is the Hamiltonian for the massive vector field in terms of A^μ and/or $F^{\mu\nu}$? Show that the Hamiltonian is positive if $m_T^2 > 0$.

(i) (15 pts) Decompose $A_\mu(x)$ in plane waves as follows:

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_{\mathbf{k}}}} \sum_{r=1}^{3} \left(a_{\mathbf{k}}^{(r)}\, \varepsilon_\mu^{(r)}(\mathbf{k})\, e^{-ik\cdot x} + a_{\mathbf{k}}^{(r)\dagger}\, \varepsilon_\mu^{(r)}(\mathbf{k})^*\, e^{ik\cdot x} \right),$$

where $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m_T^2}$.

A complete set of *three* orthogonal transverse polarization 4-vectors $\varepsilon_\mu(\mathbf{k})$ can be chosen which satisfy,

$$\varepsilon_\mu^{(r)}(\mathbf{k})\varepsilon^{\mu\,(s)}(\mathbf{k})^* = -\delta^{rs},$$

and,

$$\sum_{r=1}^{3} \varepsilon_\mu^{(r)}(\mathbf{k})\, \varepsilon_\nu^{(r)}(\mathbf{k})^* = -g_{\mu\nu} + \frac{k_\mu\, k_\nu}{m_T^2}.$$

In the rest frame find a set of orthogonal transverse polarization 4-vectors which satisfy the above relations (and show that they satisfy these relations).

(j) (15 pts) Using the equal time commutators one can show (*but you don't have to*) that $a_{\mathbf{k}}^{(r)}$ and $a_{\mathbf{k}}^{(r)\dagger}$ satisfy the harmonic oscillator commutation relations,

$$\begin{aligned} \left[a_{\mathbf{k}}^{(r)}, a_{\mathbf{k}'}^{(s)\dagger}\right] &= (2\pi)^3 \delta^3(\mathbf{k}-\mathbf{k}')\, \delta^{rs}. \\ \left[a_{\mathbf{k}}^{(r)}, a_{\mathbf{k}'}^{(s)}\right] &= \left[a_{\mathbf{k}}^{(r)\dagger}, a_{\mathbf{k}'}^{(s)\dagger}\right] = 0. \end{aligned}$$

If μ and ν are spatial indices, show by explicit calculation that

$$\langle 0|T\left(A_\mu(x) A_\nu(y)\right)|0\rangle = \int \frac{d^4k}{(2\pi)^4} \frac{-i\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{m_T^2}\right)}{k^2 - m_T^2 + i\epsilon}\, e^{-ik\cdot(x-y)}.$$

If μ and ν are time indices then derivation of the above relation is more subtle because of the time ordering, but in fact it is appropriate to use this relation for the contraction of arbitrary components of A_μ and A_ν in the calculation of Feynman diagrams in massive electrodynamics.

Decoupling of the Helicity-0 Mode:

(k) (15 pts) We can couple a conserved background current $J^\mu(x)$ with $\partial_\mu J^\mu(x) = 0$ to the massive vector field by adding to the action the term $S_I = \int d^4x A_\mu(x) J^\mu(x)$. Fourier transforming $J^\mu(x)$, define

$$J^\mu(x) = \int \frac{d^4k}{(2\pi)^4}\, \tilde{J}^\mu(k)\, e^{-ik\cdot x}.$$

Using the expansion of $A_\mu(x)$ in plane waves, show that the coupling term in the action takes the form,

$$S_I = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_{\mathbf{k}}}} \sum_{r=1}^{3} \left(a_{\mathbf{k}}^{(r)}\, \varepsilon_\mu^{(r)}(\mathbf{k})\, \tilde{J}^\mu(-\omega_{\mathbf{k}}, -\mathbf{k}) + \text{h.c.}\right).$$

(l) (15 pts) In momentum space the current conservation law becomes $k_\mu \widetilde{J}^\mu(k) = 0$. The expansion of $A_\mu(x)$ in plane waves includes modes with helicity ± 1 and 0. Consider a mode with spatial momentum in the $\mathbf{x}^3$ direction: $k^\mu = (\sqrt{k_3^2 + m_T^2}, 0, 0, k_3)$. The polarization of the normalized helicity-0 mode is then $\varepsilon_\mu^{(3)}(k_3) = \frac{1}{m_T}(k_3, 0, 0, -\sqrt{k_3^2 + m_T^2})$.

By expanding in powers of m_T/k_3 show that in the massless photon limit, $\varepsilon_\mu^{(3)}(k_3)\widetilde{J}^\mu(-\omega_{k_3}, -k_3) \to 0$. In other words, the coupling of the helicity-0 mode to the conserved current vanishes as the photon mass goes to zero.

Comments: 1) You can also calculate the Hamiltonian, spatial momentum and angular momentum of the electromagnetic field in terms of the creation and annihilation operators and obtain familiar results. For example,

$$: H := \int \frac{d^3k}{(2\pi)^3}\, \omega_{\mathbf{k}} \sum_{r=1}^{3} a_{\mathbf{k}}^{(r)\dagger} a_{\mathbf{k}}^{(r)}.$$

In your spare time you are welcome to do this exercise. 2) The carriers of the weak interactions, the W and Z bosons, are massive vector fields, and the theory you have just studied is a good starting point for a discussion the weak interactions. 3) In the literature what we have called the helicity-0 mode is unfortunately often referred to as longitudinal. This is just a warning to help avoid future confusion. 4) As a consequence of the decoupling of the helicity-0 mode, the massless limit of massive electrodynamics is equivalent to massless electrodynamics. The helicity-0 mode is still there, but it is not produced in any interactions. One could do the same for the theory of a spin-2 field, *i.e.* a symmetric tensor field $h_{\mu\nu}$ coupled to a conserved stress-energy tensor. This is the theory of linearized massive gravity. In the massless limit of this theory the helicity ± 2 modes remain coupled, as in the massless theory, and the helicity ± 1 modes decouple. However, the helicity-0 mode remains coupled. Hence, at the linearized level the massless limit of massive gravity is not equivalent to massless gravity. This fact is known as the Van Dam-Veltman-Zakharov discontinuity, and is the source of much confusion and discussion in the gravity literature.