Physics 721, Fall 2005 Josh Erlich
Problem Set 3: Quantized Scalar Fields
Due Tuesday, September 27

1. *Lorentz invariant measure*

Consider a boost in the $\hat{\mathbf{x}}_3$ direction which acts on 4-vectors as $a^\mu \to a^{\mu\prime} = \Lambda^\mu{}_\nu \, a^\nu$, where,

$$\Lambda = \begin{bmatrix} \gamma & 0 & 0 & \gamma v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma v & 0 & 0 & \gamma \end{bmatrix}, \qquad \gamma = (1-v^2)^{-1/2}.$$

Show that the volume element,

$$\frac{d^3k}{2\omega} = \frac{d^3k'}{2\omega'},$$

where $\mathbf{k}$ and $\mathbf{k}'$ are 3-momenta related by a boost in the $\hat{\mathbf{x}}_3$ direction, and ω and ω' are the associated energies. Use only the standard relation for change of volume element under a change in coordinates.

The volume element is obviously invariant under rotations, and under boosts in an arbitrary direction. Hence, it is Lorentz invariant, and when integrated against a scalar function will yield a Lorentz invariant quantity.

2. *Green's functions and time ordered products*

The time-ordered product of two fields, $\phi_1(x)$ and $\phi_2(x)$, is defined by,

$$\begin{aligned} T[\phi_1(x)\phi_2(y)] &= \phi_1(x)\phi_2(y) \ \text{ if } x_0 > y_0 \\ &= \phi_2(y)\phi_1(x) \ \text{ if } x_0 < y_0. \end{aligned}$$

Using only the field equation of motion and the equal time commutation relations, show that for a free real scalar field of mass m,

$$(\partial_\mu \partial^\mu + m^2) \, \langle 0|T[\phi(x)\phi(y)]|0\rangle = -i\delta^4(x-y).$$

Hence, $i\,\langle 0|T[\phi(x)\phi(y)]|0\rangle$ is a Green's function for the Klein-Gordon equation. It is also known as the **Feynman propagator** for the real scalar field, and will be important to us shortly.