1. Lorentz invariant measure

Consider a boost in the \( \hat{x}_3 \) direction which acts on 4-vectors as \( a^\mu \rightarrow a'^\mu = \Lambda^\mu_\nu a^\nu \), where,

\[
\Lambda = \begin{bmatrix}
\gamma & 0 & 0 & \gamma v \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\gamma v & 0 & 0 & \gamma \\
\end{bmatrix}, \quad \gamma = (1 - v^2)^{-1/2}.
\]

Show that the volume element,

\[
\frac{d^3k}{2\omega} = \frac{d^3k'}{2\omega'},
\]

where \( k \) and \( k' \) are 3-momenta related by a boost in the \( \hat{x}_3 \) direction, and \( \omega \) and \( \omega' \) are the associated energies. Use only the standard relation for change of volume element under a change in coordinates.

The volume element is obviously invariant under rotations, and under boosts in an arbitrary direction. Hence, it is Lorentz invariant, and when integrated against a scalar function will yield a Lorentz invariant quantity.

2. Green’s functions and time ordered products

The time-ordered product of two fields, \( \phi_1(x) \) and \( \phi_2(x) \), is defined by,

\[
T[\phi_1(x)\phi_2(y)] = \phi_1(x)\phi_2(y) \text{ if } x_0 > y_0 \\
= \phi_2(y)\phi_1(x) \text{ if } x_0 < y_0.
\]

Using only the field equation of motion and the equal time commutation relations, show that for a free real scalar field of mass \( m \),

\[
(\partial_\mu \partial^\mu + m^2) \langle 0|T[\phi(x)\phi(y)]|0 \rangle = -i\delta^4(x - y).
\]

Hence, \( i \langle 0|T[\phi(x)\phi(y)]|0 \rangle \) is a Green’s function for the Klein-Gordon equation. It is also known as the Feynman propagator for the real scalar field, and will be important to us shortly.