Physics 721, Fall 2005 Josh Erlich

Problem Set 2: Classical Field Theory

Due Tuesday, September 20

1. *Symmetries of the Free Complex Scalar Field*

Consider the theory of a complex scalar field ϕ with Lagrangian density,

$$\mathcal{L} = |\partial_\mu \phi|^2 - m^2 |\phi|^2.$$

This theory is symmetric under space-time translations, Lorentz transformations, and the global symmetry $\phi \to \exp[i\theta]\,\phi$, $\phi^* \to \exp[-i\theta]\,\phi^*$.

a.) Treating ϕ and ϕ^* as independent fields, what are the Euler-Lagrange equations for this theory?

b.) Construct the currents associated with space-time translations and the global phase rotation.

c.) What are the corresponding conserved charges?

d.) Decompose ϕ into Fourier modes as follows:

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_{\mathbf{k}}}} \left(a_{\mathbf{k}}\, e^{i(\mathbf{k}\cdot\mathbf{x} - \omega_{\mathbf{k}} t)} + b^\dagger_{\mathbf{k}}\, e^{-i(\mathbf{k}\cdot\mathbf{x} - \omega_{\mathbf{k}} t)} \right),$$

where $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$. When the field theory is canonically quantized, the operators $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$ become harmonic oscillator lowering operators. What are the conserved charges in terms of the operators $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$?

2. *Conserved Currents and Local Symmetry Transformations*

Suppose the action of a classical field theory is invariant under a global symmetry transformation parametrized by a continuous parameter λ. Schematically, if $\phi(x) \to \phi(x,\lambda)$ then $S \to S$ (without using the equations of motion).

a.) Consider the infinitessimal transformation $\phi(x) \to \phi(x) + \lambda\, \partial\phi/\partial\lambda|_{\lambda=0}$. Using the Euler-Lagrange equations, show that invariance of the action implies,

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \left.\frac{\partial \phi(x,\lambda)}{\partial \lambda}\right|_{\lambda=0} - F^\mu \right) = 0$$

for some set of functions $F^\mu(\phi, \partial_\mu \phi, x)$.

b.) Consider the same transformation, but allow the parameter λ to be a function of spacetime $\lambda(x)$. This is no longer a symmetry of the theory, but we can ask how the action varies under such a transformation.

Show that under the infinitessimal transformation $\phi(x) \to \phi(x) + \lambda(x)\, \partial\phi(x,\lambda)/\partial\lambda|_{\lambda=0}$, the action varies by,

$$S \to S + \int d^4x\, \partial_\mu \lambda(x) \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\partial \phi(x,\lambda)}{\partial \lambda}\bigg|_{\lambda=0} - F^\mu \right).$$

Hence the current,

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\partial \phi(x,\lambda)}{\partial \lambda}\bigg|_{\lambda=0} - F^\mu$$

appears as the variation of the action with respect to a transformation in which the symmetry transformation is made local.

3. *More Motivation for Canonical Quantization*

The **Poisson bracket** formulation of classical mechanics provides the most direct approach to quantization. If $F(q_i, p_i, t)$ and $G(q_i, p_i, t)$ are two functions of the phase space variables and time, then the Poisson bracket is defined as,

$$\{F, G\}_{PB} \equiv \sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} \right).$$

a.) Show that $\{q_i, p_j\}_{PB} = \delta_{ij},\ \{q_i, q_j\}_{PB} = \{p_i, p_j\}_{PB} = 0.$

b.) Using Hamilton's equations, show that:

$$\frac{dF}{dt} = -\{H, F\}_{PB} + \frac{\partial F}{\partial t},$$

where H is the Hamiltonian.

c.) Assume the classical functions F, G, and H are to become operators $\hat{F}$, $\hat{G}$ and $\hat{H}$ in quantum mechanics. The claim is that the Poisson bracket will then become the commutator of the corresponding operators, up to an overall constant. What overall constant a makes the replacement $\{F, G\} \to a[\hat{F}, \hat{G}]$ consistent with the commutation relations of coordinates and momenta and the Heisenberg equations of motion?