Physics 721, Fall 2005 Josh Erlich

Problem Set 1: The Dirac Equation

Due Thursday, September 8

1. *Lorentz transformation of products of Dirac spinors*

Under a Lorentz transformation,

$$\Psi(x) \to S(\Lambda)\Psi(x),$$

where the matrix $S(\Lambda)$ satisfies,

$$S^{-1} = \gamma_0\, S^\dagger\, \gamma_0, \quad S^{-1}\,\gamma^\mu\, S = \Lambda^\mu{}_\nu\,\gamma^\nu.$$

In class we demonstrated that $\overline{\Psi}\Psi$ is a Lorentz scalar, and $\overline{\Psi}\gamma^\mu\Psi$ is a Lorentz 4-vector.

a.) Show that $\overline{\Psi}\gamma^{\mu_1}\cdots\gamma^{\mu_n}\Psi$ transforms as a Lorentz tensor of rank n.

b.) Show that $(\overline{\Psi}\gamma^{\mu_1}\cdots\gamma^{\mu_{n_1}}\Psi)\,(\overline{\Psi}\gamma^{\nu_1}\cdots\gamma^{\nu_{n_2}}\Psi)$ transforms as a Lorentz tensor of rank n_1+n_2.

Hence, even though the γ^μ are just matrices and do not transform under Lorentz transformations, the transformation properties of spinor bilinears (and products of spinor bilinears) are what you would expect if you were to think of γ^μ as a 4-vector. Always keep in mind that it is really $\Psi(x)$ that is transforming, not γ^μ.

2. *Free particle solutions to the Dirac equation*

In class we found plane wave solutions to the Dirac equation in the Weyl basis for particles moving in the $\hat{\mathbf{z}}$-direction with arbitrary helicity (spin along the direction of motion). The positive helicity solution was,

$$\Psi(p) = \begin{pmatrix} \sqrt{E-p^3} \\ 0 \\ \sqrt{E+p^3} \\ 0 \end{pmatrix}.$$

Since the helicity operator commutes with rotations, a helicity eigenstate corresponding to a particle moving in an arbitrary direction can be obtained by applying two rotations to the state above.

What is the positive helicity plane wave solution to the Dirac equation corresponding to a particle moving with momentum $\mathbf{p}$ in a direction defined by (θ, ϕ) in the usual polar coordinates?

3. *Weyl and Majorana fermions*

In the Weyl representation, the gamma matrices are,

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

a.) Show that the gamma matrices in the Weyl representation satisfy the Clifford algebra, $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$.

b.) Write the 4-component Dirac spinor as,

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix},$$

where ψ_L and ψ_R each have two components and are called *Weyl spinors.*

Write the Dirac equation, $(i\hbar \not{\partial} - mc)\Psi = 0$ in terms of ψ_L and ψ_R.

c.) Show that if $m = 0$ then the Dirac equation decomposes into independent equations for ψ_L and ψ_R.

d.) Dirac argued that a spinor satisfying the Dirac equation cannot have two components because the four matrices β and α^i must mutually anticommute, and there are only three independent Hermitian anticommuting 2×2 matrices σ^i.

If $m = 0$ then you derived an equation for the two-component spinor ψ_L. How did you get around Dirac's apparent need for a four-component object?

e.) Consider an alternative equation (the Majorana equation) for a two-component spinor χ which transforms as the top two components of the Dirac spinor (ψ_L) in the Weyl basis:

$$i\hbar\, \overline{\sigma}^\mu\, \partial_\mu \chi - imc\sigma^2\, \chi^* = 0,$$

where $\overline{\sigma}^\mu = (1, -\sigma^i)$.

Show that each component of χ satisfies the Klein-Gordon equation,

$$-\hbar^2 \partial_\mu \partial^\mu \chi = m^2 c^2 \chi.$$

f.) Show that the Majorana equation is relativistically invariant.

A two-component spinor satisfying this equation is called a *Majorana* spinor. Neutrinos may be Majorana spinor fields satisfying an equation of this type.