Physics 721, Fall 2005 Josh Erlich

Midterm Exam. Due 9:30am Tuesday, October 18, 2005.

You may consult any notes you have taken, any notes I have given you, and Peskin & Schroeder's textbook. That is all you may consult, out of fairness to your classmates. You should work alone. I expect that this exam will take less than 8 hours to complete, but you are welcome to use as much time as necessary.

If you have any questions, feel free to email me at erlich@physics.wm.edu, or phone me on my cell phone at (425)876-5171. I will post answers to any questions I receive on the course website at http://physics.wm.edu/∼erlich

There is one problem on the exam, but it is a long one. The problem has been divided into 11 parts.

The problem is based on the Lagrangian density,

$$\mathcal{L} = \pm\overline{\psi}\left(i\partial\!\!\!/ - \widetilde{m} + im\,\gamma^5\right)\psi,$$

where $\psi(x)$ is a 4-component Dirac spinor field.

1) Show that the action $\int d^4x\,\mathcal{L}$ is real (or Hermitian if $\psi(x)$ is an operator).
(5 points)

2) Show that $\mathcal{L}$ is invariant under Lorentz transformations connected to the identity if $\psi(x)$ transforms like a Dirac spinor. (These are the usual Lorentz transformations studied in class. They do not involve space-time reflections.)
(5 points)

3) Find the Euler-Lagrange equations for this theory, treating ψ and $\overline{\psi}$ as independent fields. Check that the equations are consistent with one another.
(5 points)

4) Show that each component of $\psi(x)$ satisfies the Klein-Gordon equation. What is the squared mass of the field ψ? (5 points)

5) Calculate the Hamiltonian H in terms of ψ and its derivatives. (You can simplify your expression by using the Euler-Lagrange equations if you like.)
(10 points)

6) Calculate the spatial momentum $\mathbf{P}$ in terms of ψ and its derivatives.
(10 points)

7) Calculate the $\mathbf{x^3}$-component of the angular momentum $\mathbf{J}^z$ in terms of ψ and its derivatives. (10 points)

8) What is the conserved current due to the symmetry $\psi \to e^{-i\theta}\psi$, where θ is a real parameter independent of x^μ? What is the corresponding charge Q in terms of ψ and its derivatives? (10 points)

For the rest of the problem, assume $\widetilde{m} = 0$.

9) Canonically quantize the theory using anticommutation relations, *i.e.* decompose $\psi(x)$ in a complete set of solutions of the Euler-Lagrange equations and determine the appropriate anticommutation relations of the Fourier components of $\psi(x)$. (20 points)

Hint: You will need to find the plane wave solutions to the Euler-Lagrange equations for this theory. You will also need to derive new completeness and orthogonality relations. These are a straightforward generalization of the analogous relations for the Dirac field as derived in class.

10) Calculate the Hamiltonian H in terms of the Fourier coefficients. Leave your result in terms of an integral over a single factor of d^3k.

If a consistent sign choice exists, use your result to fix the overall sign of the Lagrangian by insisting that the Hamiltonian be bounded below. What are the defining properties of your vacuum state $|0\rangle$? What is the properly normal ordered form of the Hamiltonian? (15 points)

11) Calculate the commutator of H with the Fourier coefficients of $\psi(x)$ and their Hermitian conjugates (*i.e.* $[H, a_{\mathbf{k}}^{r\,\dagger}]$, *etc.* in a hopefully obvious notation). What does this tell you about the states created by the Fourier coefficients or their conjugates acting on the vacuum? (5 points)

You can convince yourself that $\mathbf{P}$, Q and $\mathbf{J}^z$ will work out similarly. How does the field theory defined by $\mathcal{L}$ compare with the theory defined by the Dirac Lagrangian density, $\mathcal{L}_D = \overline{\psi}\,(i\partial\!\!\!/ - m)\,\psi$? (For future thought: We haven't studied discrete symmetries yet, but the new mass term is odd under parity and time reversal because of the γ^5.)

Possibly useful information:

The Minkowski metric is,

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

The 4×4 gamma matrices in the Weyl basis are,

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = (\gamma^5)^\dagger = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\overline{\psi} = \psi^\dagger \gamma^0, \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad (\gamma^0)^\dagger = \gamma^0, \quad (\gamma^0\gamma^\mu)^\dagger = \gamma^0\gamma^\mu.$$

$$(\gamma^5)^2 = 1, \quad \{\gamma^5, \gamma^\mu\} = 0.$$

Under a Lorentz transformation $x^\mu \to \Lambda^\mu{}_\nu \, x^\nu$, the Dirac field transforms as:

$$\psi(x) \to \psi'(x) = S(\Lambda)\, \psi(\Lambda^{-1}x),$$

where,

$$S\left(\Lambda\left(\omega\right)\right) = \exp\left[-\frac{i}{4}\omega_{\mu\nu}\, \sigma^{\mu\nu}\right],$$

and,

$$\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right].$$

In the Weyl basis,

$$\sigma^{ij} = \epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}, \quad \sigma^{0k} = -i \begin{pmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{pmatrix},$$

where ϵ^{ijk} is totally antisymmetric in i, j, k and $\epsilon^{123} = 1$.

For an infinitessimal counterclockwise rotation by angle θ about the $\mathbf{x}^3$-axis, $\omega_{12} = -\omega_{21} = \theta$.

For an infinitessimal boost by velocity β in the $\mathbf{x}^3$-axis, $\omega_{03} = -\omega_{30} = \beta$.