Physics 721, Fall 2005 Josh Erlich

Final Exam. Due 9:30am Tuesday, November 22, 2005.

You may consult any notes you have taken, any notes I have given you, and Peskin & Schroeder's textbook. That is all you may consult, out of fairness to your classmates. You should work alone.

If you have any questions, feel free to email me at erlich@physics.wm.edu, or phone me on my cell phone at (425)876-5171. I will post answers to any questions I receive on the course website at http://physics.wm.edu/∼erlich

There are two problems on the exam.

1. *Pseudoscalar Mesons* (50 points)

We have discussed in some detail Yukawa's theory of meson-nucleon ineractions, with interaction Hamiltonian density $\mathcal{H}_I = g\,\overline{N}N\phi$. In Yukawa's theory the meson is a scalar field. Actual pions are pseudoscalar, meaning they change sign under a parity transformation. They couple to nucleons with $\mathcal{H}_I = g\,\overline{N}(i\gamma^5)N\phi$. In this problem you will study an improved model of nucleon-pion interactions, and contrast it with Yukawa's theory.

The Lagrangian density describing the system of neutrons and neutral pions is,

$$\mathcal{L} = \overline{N}\,(i\partial\!\!\!/ - m)\,N + \frac{1}{2}(\partial_\nu\phi)^2 - \frac{1}{2}\mu^2\phi^2 - g\,\overline{N}(i\gamma^5)N\phi,$$

where the neutron N is a 4-component Dirac spinor field and the pion ϕ is a real pseudoscalar. (You will not need to use the fact that ϕ changes sign under parity, but the form of $\mathcal{H}_I$ dictates that it must do so.)

Consider elastic scattering of neutrons $N + N \to N + N$.

a) Calculate the scattering matrix element for this process at $\mathcal{O}(g^2)$. Draw the Feynman diagrams and label the quantum numbers of each neutron.

b) Show that the amplitude vanishes if either the ingoing neutrons have the same quantum numbers, or the outgoing neutrons have the same quantum numbers. (This had to be the case by Pauli's exclusion principle.)

c) Take the nonrelativistic limit of the scattering matrix element, *i.e.* assume the spatial momenta of the neutrons are small compared to m. Use the appropriate solutions of the Dirac equation in the nonrelativistic limit, and express the result in terms of momenta $\mathbf{p}$ and spins $\mathbf{S}$ (or Pauli matrices $\boldsymbol{\sigma}$).

d) In the center of mass frame we can think of this process as scattering of a nucleon off of the pion potential due to the other nucleon. In the Born approximation of nonrelativistic quantum mechanics for spin-1/2 particles,

$$\langle \mathbf{p}', r'|(S-1)|\mathbf{p}, r\rangle = -i\xi^{(r')\,\dagger}\,\widetilde{V}(\mathbf{S}, \mathbf{S}_{\text{source}}; \mathbf{p}', \mathbf{p})\,\xi^{(r)}\,(2\pi)\,\delta(\omega_{\mathbf{p}'} - \omega_{\mathbf{p}}),$$

where $\xi^{(r)}$ and $\xi^{(r')}$ are 2-component Pauli spinors describing the initial and final state of the scattered nucleon; $\mathbf{S} = \boldsymbol{\sigma}/2$ is the spin operator for the scattered nucleon; and $\mathbf{S}_{\text{source}} = \xi^\dagger_{\text{source}}(\boldsymbol{\sigma}/2)\xi_{\text{source}}$ is the spin matrix element of source, assumed to be held fixed so that the initial and final spins are the same.

What is the momentum-space potential $\widetilde{V}(\mathbf{S}, \mathbf{S}_{\text{source}}; \mathbf{p}', \mathbf{p})$ due to the pion field of a heavy nucleon with spin $\mathbf{S}_{\text{source}}$? Identify the direct and exchange potentials.

e) Fourier transform the direct potential with respect to the momentum transfer to determine the position space potential $V(|\mathbf{x}|)$ as a function of the distance between the neutrons, $|\mathbf{x}|$.

When Fourier transforming to position space, factors of momenta in the numerator become derivatives in position space, as usual. Assume that $|\mathbf{x}| \neq 0$ and take any derivatives appearing in your expression for the potential. You should leave your answer in terms of the $2{\times}2$ spin matrix $\mathbf{S}$ for the scattered neutron, the spin matrix element $\mathbf{S}_{\text{source}}$ for the neutron source. the distance between neutrons $|\mathbf{x}|$, and the parameters of the Lagrangian.

Does the potential decrease exponentially with distance as in Yukawa's theory?

f) Consider scattering of unpolarized beams of *different* nucleons due to neutral pion exchange. In other words, consider $N_1 + N_2 \to N_1 + N_2$ in the theory described by,

$$\begin{aligned}\mathcal{L} &= \overline{N}_1 \left(i\partial\!\!\!/ - m_1\right) N_1 + \overline{N}_2 \left(i\partial\!\!\!/ - m_2\right) N_2 \\ &\quad + \frac{1}{2}(\partial_\nu \phi)^2 - \frac{1}{2}\mu^2\phi^2 - g\left(\overline{N}_1(i\gamma^5)N_1\phi + \overline{N}_2(i\gamma^5)N_2\phi\right).\end{aligned}$$

Calculate the relativistic differential cross section in the center of mass frame, at leading order in g, averaged over initial spins and summed over final spins. Perform all required traces of gamma matrices. You can leave your answer in terms of the COM momenta, *i.e.* you do not need to work through the relativistic kinematics to relate initial and final momenta.

Hints: 1) In considering the nonrelativistic limit, it is probably easier to use the Dirac basis for the γ matrices and for the solutions to the Dirac equation. 2) You should write the 4-component Dirac spinors in terms of the 2-component spinors that appear in Schrödinger's equation for a spin-1/2 particle. Recall how we showed that Dirac's theory predicts the gyromagnetic ratio of the electron. 3) In the last part of the problem the nucleons are distinguishable, so only one Feynman diagram contributes at lowest order in g.

2. *Mott/Rutherford scattering* (50 points)

In this problem you will consider scattering of electrons off of the Coulomb potential of a heavy nucleus. For a relativistic electron this is known as Mott scattering; in the nonrelativistic limit it is Rutherford scattering.

The electric field of the nucleus is treated as a classical, unquantized, background field for this problem. The electromagnetic interaction of the electrons is otherwise precisely that of QED:

$$H_I = e \int d^3x \, \overline{\psi} \gamma^\mu \psi A_\mu.$$

We can choose the 4-vector potential A^μ to be:

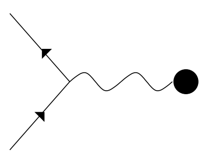
$$A^0 = \frac{-Ze}{4\pi|\mathbf{x}|}, \qquad \mathbf{A} = 0.$$

In momentum space,

$$\int d^4x \, e^{i(p-p')\cdot x} A^0(|\mathbf{x}|) = (2\pi)\,\delta(\omega_{\mathbf{p}} - \omega_{\mathbf{p}'})\,\frac{(-Ze)}{|\mathbf{p} - \mathbf{p}'|^2}.$$

a) Define $\langle \mathbf{p}', r'|(S-1)|\mathbf{p}, r\rangle \equiv i\mathcal{M}\,(2\pi)\,\delta(\omega_{\mathbf{p}'} - \omega_{\mathbf{p}})$.

Show that the vertex appearing in the Feynman rules for $i\mathcal{M}$ is,



$$= -ie\gamma^\mu \widetilde{A}_\mu(\mathbf{q}),$$

where $\widetilde{A}_\mu(\mathbf{q})$ is the three dimensional Fourier transform of $A_\mu(x)$.

Calculate the lowest order amplitude for an electron of momentum $\mathbf{p}$ and spin r to scatter off of the Coulomb field into a final state electron with momentum $\mathbf{p}'$ and spin r'.

b) Show that the differential cross section is,

$$d\sigma = \frac{1}{v\,(2\omega_{\mathbf{p}})(2\omega_{\mathbf{p}'})}\,\frac{d^3p'}{(2\pi)^3}\,|\mathcal{M}|^2\,(2\pi)\delta(\omega_{\mathbf{p}'} - \omega_{\mathbf{p}}),$$

where v is the speed of the incoming electron with respect to the static Coulomb source. Integrate over $|\mathbf{p}'|$ using the energy-conserving delta function to find a simple expression for $d\sigma/d\Omega$.

c) What is the differential cross section for scattering of unpolarized electrons at lowest order in e? Perform the average over initial spins and sum over final spins.

d) Recover the Rutherford cross section in the nonrelativistic limit $v \to 0$:

$$\frac{d\sigma}{d\Omega} = \frac{(Z\alpha)^2}{4m^2\, v^4\, \sin^4(\theta/2)},$$

where $\alpha = e^2/(4\pi)$, m is the electron mass, v is the electron speed, and θ is the scattering angle.

e) Now consider scattering of an electron with definite helicity $\mathbf{S} \cdot \hat{\mathbf{p}}$. Choose the $\mathbf{x}_3$-axis as the initial direction of motion, and assume the helicity is initially positive, *i.e.* the spin is along the direction of motion. Calculate the probability that the helicity remains unchanged after scattering off of the nucleus.

f) Do the same assuming the initial spin is opposite the direction of motion.

Hints: 1) In the nonrelativistic limit the spin of the electron should not change after scattering, so the probability that the spin will continue to point along the direction of motion in that limit is $\cos^2(\theta/2)$. 2) In the extreme relativistic limit you should find instead that the helicity is unchanged in the scattering process.

Possibly useful information:

The Minkowski metric is,

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

The 4×4 gamma matrices in the Weyl basis are,

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = (\gamma^5)^\dagger = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The 4×4 gamma matrices in the Dirac basis are,

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = (\gamma^5)^\dagger = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In any basis,

$$\overline{\psi} = \psi^\dagger \gamma^0, \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad (\gamma^0)^\dagger = \gamma^0, \quad (\gamma^0\gamma^\mu)^\dagger = \gamma^0\gamma^\mu.$$

$$(\gamma^5)^2 = 1, \quad \{\gamma^5, \gamma^\mu\} = 0.$$

$$\overline{\gamma^\mu} = \gamma^0(\gamma^\mu)^\dagger\gamma^0 = \gamma^\mu, \quad \overline{\gamma^5} = \gamma^0(\gamma^5)^\dagger\gamma^0 = -\gamma^5.$$