Physics 690/482-02, Spring 2005 Josh Erlich

Problem Set 5: Gauge Invariance

This is a short problem set just to make sure you're comfortable with gauge invariance, and it will be due this Friday, Feb 25, 2005.

1. *Gauge invariant Lagrangians*

Consider the following Lagrangian density for a *complex* scalar field, $\psi(x)$, in $d+1$ dimensions:

$$\mathcal{L} = |(\partial_\mu - iq\, A_\mu)\psi|^2 \ .$$

Show that $\mathcal{L}$ is invariant under the following gauge transformation:

$$\begin{aligned} A_\mu &\rightarrow A_\mu + \partial_\mu \alpha(x) \\ \psi &\rightarrow \psi\, e^{iq\,\alpha(x)} \end{aligned}$$

The derivative $D_\mu \psi \equiv (\partial_\mu - iq\, A_\mu)\psi$ is called the **gauge covariant derivative**, and the parameter q is referred to as the **charge** of the field ψ.

2. *Massless scalars from a gauge field on a torus*

Consider $d+1$ dimensional E&M compactified on the n-dimensional torus T^n.

- How many massless scalars are there in the $d+1-n$ dimensional Kaluza-Klein decomposition?
- The Lagrangian is invariant under $A_\mu \rightarrow A_\mu + \partial_\mu \alpha$. If the coordinates in the n compact dimensions are $y_1, \ldots, y_n$, what properties do the gauge function $\alpha(y_1, \ldots, y_n)$ have to satisfy to be consistent with the boundary conditions?

3. *Gauge invariance in curved spacetime*

The generally covariant form of the E&M Lagrangian density is,

$$\mathcal{L} = -\frac{1}{4}\sqrt{g}F_{\mu\nu}F_{\sigma\rho}\, g^{\mu\sigma}g^{\nu\rho} \ ,$$

where,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

Show that the Lagrangian is invariant under the transformation,

$$A_\mu \to A_\mu + \partial_\mu \alpha(x).$$

This should require no more than two lines to prove. Hence, you have demonstrated that gauge invariance of pure electrodynamics persists in curved spacetime. Also notice that the definition of the field strength $F_{\mu\nu}$ is the same in curved spacetimes as in flat spacetime. This is because $D_\mu A_\nu - D_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$, where D_μ is the generally covariant derivative.